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Proof of Frankl's Conjecture: A Non-Constructive Approach

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Abstract: Let U be a finite set and $\mathscr{X}$ a family of nonempty subsets of U, which is closed under unions. We establish a connection between Frankl's conjecture and equipollence sets, in which a complementary set is an Equipollence set on the Frobenius group. We complete the proof of the union-closed sets using a non-constructive approach. The proof relies upon that we need to prove, that the series of the prime divisor diverges, and there exists x_i which appears at least half distributed in subsets.

Keywords: Frankl's Conjecture, Union-Closed, Extremal Set Theory, Complementary Sets, Primes

Introduction

In 1979, Frankl (1995) stated the conjecture, in terms of intersection-closed set families, and so the conjecture is usually credited to him and sometimes called the Frankl conjecture. It says that there is an element of U which is in at least half the sets of $\mathscr{X}$. Frankl's conjecture is one of the most popular combinatorics problems, most authors today do extensive research on this problem, and the amount of papers published is quite large. The previous best results were solutions in the properties of counterexamples due to Dohmen (2001); in the graph formulation due to El-Zahar (1997), Bruhn *et al*. (2015); in lattice version due to Abe and Nakano (1998), Abe (2000). Marković (2007) proves this when $|\bigcup \mathscr{X}| \leq 10$. Bošnjak and Marković (2008) improved the bound to 11. For some historical remarks see (Winkler, 1987; Wójcik, 1992; Morris, 2006; Roberts and Simpson, 2010; Bruhn and Schaudt, 2015), etc. The Frankl conjecture has been proved for many special cases. The famous examples are the families of at most 36 sets (by Faro (1994), FC-families (introduced by Poonen (1992) and further studied by Gao and Yu (1998), Vaughan (2002; 2003; 2004). Joshi and Waphare (2019) prove Frankl's conjecture for an upper semimodular lattice.

Yet by and large, the Frankl conjecture is concerned with arrangements of the objects of a set into patterns satisfying specified rules. Difference sets and difference families, including the complementary sets and the complementary families, is one of the important objects in the theory of combinatorial designs.

In this study, we new study offers novel insights into the link between union-closed families and the complementary families, and we introduce a complementary set to construct one-one mapping and by equipollence sets and number theory to Frankl conjecture. In the end, we propose a quasi-Randomness Block Design (abbreviated q-RBD) that, we hoped, aroused many researchers' (e.g., mathematician-statisticians) great interest.

Frankl's Conjecture and its Examples

A finite non-empty family of finite sets $\mathscr{X}$ is called union-closed if, for all $X, Y \in \mathscr{X}$ implies $X \cup Y \in \mathscr{X}$, the conjecture follows.

Conjecture (Frankl) 2.1

Let $\mathscr{X}$ be a finite family of finite sets, not all empty, that is closed under taking unions. Then there exists $x \in \bigcup_{X \in \mathscr{X}} X$ such that x is an element of at least half the members of $\mathscr{X}$.

We show how Frankl's conjecture can be used in the following example.

Example 2.2

Figure 1 shows an example of a union-closed family, where we have omitted commas and parentheses for readability. There, one may count that the elements 1, 2, and 3 appear each in only 12 of the 25 member sets, which is less than half of the sets. Each of the other elements 4, 5, and 6 however is contained in 16 member sets, more than enough for the family to satisfy the conjecture.

Example 2.3

Figure 2 shows another example of a union-closed family, and there the conjecture is tight in power sets, i.e. every element appears in exactly half of the member sets. There, each element of {1, 2, 3} is contained in only 13 out of 25 member sets.

In the examples above, we now outline averaging because of its ability to shift or compress. The average set size is:

$$\frac{1}{|U|}\sum_{X\in|\mathscr{X}|}|X|\geq\frac{1}{2}|\mathscr{X}|$$

or:

$$\frac{1}{|\mathscr{X}|}\sum_{X\in|\mathscr{X}|}|X|\geq\frac{1}{2}|U|$$

$\mathscr{X}$ satisfies the conjecture.

But averaging does not always work. As usual, conjecture is proved without looking for abundant elements.

Equipollence Sets and Proof

An equipollence set is simply a set with an equal number of elements. To be equivalent, the sets should have the same cardinality. This means that there should be a one-to-one correspondence between elements of both sets.

In a union-closed family of sets $\mathscr{S}$, we display this one-to-one correspondence (e.g., $x' \leftrightarrow x$, $p_i \leftrightarrow x_i$), the equipollence, $\mathscr{X}' \sim \mathscr{X}$ and the isomorphism $\mathscr{X}' \cong \mathscr{X}$ in Fig. 3.

We give a general definition now.

Definition 3.1

Let $\mathscr{S}$ be a Frobenius group. Assume that for each $\mathscr{X}' \in \mathscr{S}$ Such that:

$$\mathscr{X} \bigcap {}^{g}\mathscr{X}' = \{1\}, \tag{1}$$

and $\forall g \in \mathscr{S} - \mathscr{X}'$. Then we say that $\mathscr{X}'$ is a Frobenius complimentary of $\mathscr{S}$.

We can now conclude that $|\mathscr{X}'| \leq |\mathscr{X}|$. Our next goal is to prove the Frankl conjecture.

Proof of Conjecture (Frankl) 2.1

Let N be a set of natural numbers, let:

$$\mathscr{X} = \{x_1, x_2, ..., x_n\} \tag{2}$$

If i and j are arbitrary. Then:

$$x_i \cap x_j \in \mathscr{X} \tag{3}$$

By Definition 3.1, we consider a Frobenius complementary set $\mathscr{X}$, the union-closed sets defined as:

$$\mathscr{X}' := \{x_1', x_2', ... \, x_n'\} \tag{4}$$

Since:

$$x_i \cap x_j \neq \phi \; (i \neq j), \tag{5}$$

and:

$$x_i \not\subseteq x_j \quad with \quad x_i \not\supseteq x_j, \tag{6}$$

we have:

$$x_i \not\subseteq x_j \;\; with \;\; x_i \not\supseteq x_j', \tag{7}$$

and:

123456
12345 12346 12356 12456 13456 23456
1234 1235 1236 1456 2456 3456
123 145 246 356 456
45 46 56
4 5 6
∅

Fig. 1: A union-closed family

123456
12356 12346 12345
1236 1235 1234
456 236 135 124 123
56 46 45 23 13 12
6 5 4 3 2 1
∅

Fig. 2: An intersection-closed family

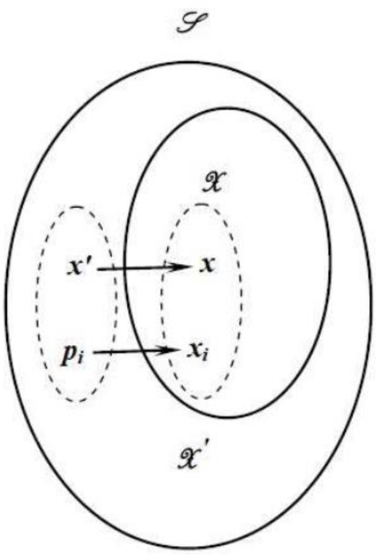

Fig. 3: Schematic view of the algebraic structure of a union-closed family

$$x_i \nsubseteq x_j \ \ with \ \ x_i \nsupseteq x_j^{'}, \tag{8}$$

means that:

$$x_i \bigcup x_j^{'} \neq \phi (i \neq j) \tag{9}$$

This means that $\mathscr{X}$ does not contain $\mathscr{X}^{'}$, and vice versa.

Since:

$$x_i \bigcap x_j \neq N, \tag{10}$$

we have:

$$x_i^{'} \bigcap x_j^{'} \neq \phi \ \ (i \neq j) \tag{11}$$

Now, let $x_1^{'}, x_2^{'}, \ldots, x_n^{'}$ are primes. By equivalent principle, we consider the finite union-closed set $\mathscr{S}$ in a family of sets, and we define:

$$\mathscr{S} := \{x_1, x_2, \ldots, x_n, x_1^{'}, x_2^{'}, \ldots, x_j^{'}\} \tag{12}$$

can be divided into two families of sets.

Note that if $\mathscr{X}$ is an open set and $\mathscr{X}^{'}$ is a Frobenius complementary set $\mathscr{X}$, then $\mathscr{S}$ is a finite union-closed set. Frankl's conjecture has a number theory form that corresponds to one another.

Next, proof by contradiction:

Assume that $\sum_{p \in P} \frac{1}{p}$ is convergent. Then there exists a natural number k such that:

$$\sum_{i \geq k+1} \frac{1}{p} < \frac{1}{2} \tag{13}$$

In this case, we certainly have:

$$\sum_{i \geq k+1} \frac{N}{p} < \frac{N}{2} \tag{14}$$

Let T' denote the number of positive integer n, satisfies:

$$n \leq N, \tag{15}$$

and at least T' is divisible by a large number p_{k+1}. We have:

$$T' \leq \sum_{i \geq k+1} \left\lfloor \frac{N}{p_i} \right\rfloor < \frac{N}{2} \tag{16}$$

Let T denote the number of positive integer n.

Since:

$$N = T + T' \tag{17}$$

Then we certainly have:

$$T \leq \frac{N}{2} \tag{18}$$

As required, let $n = N$, where the T is an element:

$$x \in \bigcup \mathscr{S} \tag{19}$$

We have:

$$T > \frac{N}{2}, \tag{20}$$

and we have our desired contradiction and in violation of the union-closed sets conjecture. Since by Eq. (17), This implies that it would have:

$$T = \frac{N}{2} \tag{21}$$

which proves that for each T, there is one, and only one, prime divisor p which maps to it. We have our desired contradiction.

So where are the singletons? We shall confirm below.

By all the natural numbers N are sets. We define:

$$T(x) := x\{x\} \tag{22}$$

is an ordinal, we have:

$$x < T(x) \tag{23}$$

$\mathscr{S}$ is always a (finite) union-closed family and contains all singletons of S. We define:

$$U := \bigcup_{T \in \mathscr{S}} T \tag{24}$$

is the universe. By Eq. (21), we see that only x lies in $\frac{1}{2}\bigcup \mathscr{S}$, which a finite set U of each element with a hidden numeric structure, in the case of a lot of p, we have $|\mathscr{X}^{'}| = |\mathscr{X}|$, as shown in Fig. 3. These forces:

$$\lceil x \rceil = \left\lceil \frac{n}{2} \right\rceil = \lceil p \rceil \tag{25}$$

We see that x satisfies the union-closed sets conjecture.

Concluding Remarks

In this study, we presented a non-constructive approach, which offers rigorous proof of Frankl's conjecture. It is a new generating permutation problem for the Frobenius group, in a sense. But the focus is that we hope the equipollence sets will act as a research object of design theory. Interestingly, it is a hidden numeric structure. We find, may involve the quasi-randomness block design, abbreviated q-RBD, it is also some potential applications involved.

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Ethics

The author confirms that this article is original and contains unpublished material. The author has read and approved the manuscript and no ethical issues are involved.

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