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Modification of the Ornstein Uhlenbeck Process to Incorporate the Influence of Speculation on Volatility in Financial Markets

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Abstract: Financial market participants often speculate on how markets would behave in the light of certain information at hand. This speculation contributes to volatility within the financial market and consequently, it makes the market unstable. The Ornstein Uhlenbeck (OU) model has intensively been used in modelling volatility, however, the contribution of speculation on volatility has not been studied in the OU model. Therefore, this study focuses on the modification of the OU model by incorporating a time dependent exponential function that caters for the contribution of speculation on volatility. The statistical properties of the Improved OU model are then studied and the results compared with properties of the OU model. NAD/USD exchange rate data is used to compare and validate the Improved model with the OU model. It was found that both the OU and Improved OU model had a similar expected price, while variance of price for the OU model stabilised upwards up to 16 and variance of price for the Improved OU model stabilised downwards up to 0.01. The variance of the Improved model was found to be much lower than that of the OU model. Additionally, it was found that the distribution of the forecasted price changed with different lead times for the OU model whereas, the distribution of the forecasted price for the Improved OU model did not change with different lead times. Thus, the OU model is a time specific model whereas the Improved OU model is an invariant time model. Consequently, the Improved OU model was found to be more efficient than the OU model.

Keywords: Volatility, Speculation, Exchange Rate, Ornstein Unlenbeck

Introduction

According to Hirota *et al*. (2018), speculators are short term partakers in financial markets. When markets are occupied by speculators, prices can be susceptible to excess volatility (Keynes (1936), Shiller (2000), Stiglitz (1989). The role of speculators in financial markets has been a source of considerable interest and controversy in recent years. An example of the effects of speculation is given in an article by Provan *et al*. (2020), where a rumour of positive news regarding a Covid-19 vaccine that was developed at the University of Oxford increased shares in Astra Zeneca by 5 percent. To understand the effect such and other types of speculation has on volatility, it is important to model volatility and investigate what effect speculation has on volatility using the same model. Geometric Brownian Motion (GBM) has been used widely in modelling price volatility since its introduction in the 1960's. According to stochastic process S_t is said to follow a geometric brownian motion if it satisfies the following stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{1}$$

where, S_t is price, dS_t is the change in price, dt is the change in time and dW_t is change in a Wiener process (brownian motion). Parameters μ and σ are interpreted as the drift and the volatility term respectively. One of the drawbacks of the GBM model is that its predicted price can continue to increase to unrealistic values or decrease to generate negative price Sigman (2006), which does not depict a real-life scenario. Therefore, the Ornstein Uhlenbeck (OU) Model tries to solve this weakness by incorporating a mean reverting parameter k as shown in (2):

$$dS_t = k(\mu - S_t)dt + \sigma dW_t, \tag{2}$$

where the parameter μ, σ and k represent mean, volatility term and the rate by which the price reverts towards the mean respectively, Doob (1942). The process in (2) fluctuates randomly but tends to revert to the mean μ with the help of the mean reverting parameter k where, the behaviour of this reversion depends on both the short-term standard deviation σ and the speed of the reversion parameter k (Franco 2015). The larger the value of k, the greater the speed of mean-reversion and vice-versa (Manzoor (2015).

Although the OU model solves the weakness of the GBM model, it carries its own weakness which is the inability to predict accurate price when speculation is present in a market. Having identified the short comings of the GBM model and OU model, this study modifies the existing OU model in (2) by incorporating a speculation function to make the volatility time dependent and assesses the contribution of speculation on volatility. Secondary data obtained from a foreign exchange platform called Reuters, officially known as Refinitv Eikon (Haycock 2008) for a time period from 1993/11/24 to 2021/03/09, was used for numerical simulation. The data was cleaned and analysed using various statistical methods and software (e.g., R, EXCELL, MAPO and MATLAB).

Development of the Models

Statistical Properties of Ornstein Uhlenbeck Models

To solve for S_T, we rearrange (2) to get (3) below:

$$dS_t + kS_t dt = k\mu dt + \sigma dW_t \tag{3}$$

Multiplying both sides of (3) by the integrating factor e^{kt} and applying the product rule on the left-hand side (Gatto (2002)), we obtain

$$\frac{d}{dt}\left(e^{kt}S\right) = k\mu e^{kt}dt + \sigma e^{kt}dWt \tag{4}$$

Integrating from 0 to T and evaluating the integrals, we have:

$$e^{kT}S_T - e^0 S_0 = k\mu\frac{e^{kT} - e^{k0}}{k} + \int_0^T \sigma e^{kt}dW_t \tag{5}$$

Multiplying by e^{-kT} to isolate S_T and moving S_0 to the right, we get the solution:

$$S_T = S_0 e^{-kT} +_{\mu}\left(1 - e^{-kT}\right) +_{\sigma}\int_0^T e^{-k(T-t)}dW_t \tag{6}$$

S_T is normally distributed since the integral of a deterministic function with respect to Brownian is Gaussian (Lebovits (2014), Shahnazi *et al*. (2021); The expectation of S_T is given by (7):

$$E\left(S_T\right) = E[S_0 e^{-kT} + \mu\left(1 - e^{-kT}\right) + \sigma\int_e^T e^{-k(T-t)}dW_t\ ] \tag{7}$$

Noting that the expected value of a deterministic function with respect to Brownian is 0 (Vardar-Acar and Bulut (2015), we get the formula for the mean:

$$E\left(S_T\right) = S_0 e^{-kT} + \mu\left(1 - e^{-kT}\right) \tag{8}$$

The variance of S_T is given by (9):

$$\begin{aligned}
V\left(S_T\right) &= E\left[S_T - E\left(S_T\right)\right]^2 \\
&= E\ \left[\left(S_0 e^{-kT} + \mu\left(1 - e^{-kT}\right) + \sigma\int_0^T e^{-k(T-t)}dW_t\ \right)^2\right. \\
&\left(-\left(S_0 e^{-kT} + \mu\right)\ \left(1 - e^{-kT}\right)\right] \\
&= E\left[\sigma\int_0^T e^{-k(T-t)}dW_t\ \right]^2 \\
&= E\left[\sigma\int_0^T e^{-k(T-t)}dW_t\ \right]^2 \\
&= E\left[\sigma^2\frac{e^{-2k(T-T)}}{2k}\right]\Big\|_0^T \\
&= E\left[\sigma^2\frac{e^{-2k(T-T)}}{2k} - \frac{e^{-2k(T-0)}}{2k}\ \right] \\
&= E\left[\frac{\sigma^2}{2k}\left(1 - e^{-2kT}\right)\right] \\
&= \frac{\sigma^2}{2k}\left(1 - e^{-2kT}\right)
\end{aligned} \tag{9}$$

Thus, $ST \sim N\left((S_0^{e\text{-}kT} + \mu(1\text{-}^{e\text{-}kT}),\ \frac{\sigma^2}{2k}\left(1 - e^{-kt}\right)\right)$. Consider time T and time R where $R<T$. The covariance between S_T and S_R is given by:

$$\begin{aligned}
Cov\left(S_T, S_R\right) &= E[(S_T - E[S_T])(S_R - E[S_R])] \\
&= E\left[S_0 e^{-kT} + \mu\left(1 - e^{-kT}\right) + \sigma\int_0^T e^{-k(T-t)}dW_t\ ) - S_0 e - kT + \mu\left(1 - e^{-kT}\right)\right. \\
&(S_0 e^{-kR} + \mu\left(1 - e^{-kR}\right) + \sigma\int_0^R e^{-k(R-u)}dW_u\ ) - S_0 e^{-kR} + \mu\left(1 - e^{-kR}\right) \\
&= E\left[\sigma\int_0^T e^{-k(T-t)}\,dW_t\ \sigma\int_0^R e^{-k(R-u)}dW_u\ \right] \\
&= \sigma^2 e^{-k}\,(R+T)E\left[\int_0^T e^{kt}dW_t\ \sigma\int_0^R e^{ku}\ dW_u\ \right]
\end{aligned} \tag{10}$$

Noting that the covariance over the non-overlapping period is 0 Frankland *et al*. (2019) and applying Ito's isometry Takahiko (2014), we are left with a deterministic integral:

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$$
\begin{aligned}
Cov(S_T, S_R) &= \sigma^2 e^{-k(R+T)} \int_0^R e^{2ku} dW_u \\
&= \sigma^2 e^{-k(R+T)} + \frac{e^{2kR}-1}{2k} \\
&= \frac{\sigma^2}{2k}\left(e^{-k(R+T)+2kR} - e^{-k(R+T)}\right) \\
&= \frac{\sigma^2}{2k}\left(e^{-k(T-R)} - e^{-k(T+R)}\right)
\end{aligned}
\tag{11}
$$

Therefore:

$$Cov(S_T, S_R) = \frac{\sigma^2}{2k}\left(e^{-k(T-R)} - e^{-k(T+R)}\right) \tag{12}$$

To understand how the expected value and variance of S_T will behave in future, we revisit the limiting distributions of S_T:

$$
\begin{aligned}
\lim_{T\to\infty} E(S_T) &= \lim_{T\to\infty}\left(S_0 e^{-kT} + \mu\left(1^{-e-kT}\right)\right) \\
&= S_0 \lim_{T\to\infty} e^{-kT} + \mu\left(1 - \lim_{T\to\infty} e^{-kT}\right) \\
&= \mu\Re
\end{aligned}
\tag{13}
$$

And:

$$
\begin{aligned}
\lim_{T\to\infty} V(S_T) &= \lim_{T\to\infty}\left(\frac{\sigma^2}{2k}\left(1-e^{-2kT}\right)\right) \\
&= \frac{\sigma^2}{2k}\left(1 - \lim_{T\to\infty} e^{-2kT}\right) \\
&= \frac{\sigma^2}{2k}
\end{aligned}
\tag{14}
$$

The variance is inversely proportional to the speed of mean reversion Tsou (2011). This can be explained by the fact that, the higher the speed of mean reversion, the higher the drift towards the mean, which means lower variance.

Development of the Improved OU Models

This study proposed a model that modifies the current OU model by considering σ as a time varying function $f(t)$ as given in (15):

$$dS_t = k(\mu - S_t)dt + f(t)dW_t, \tag{15}$$

where, S_t is price, dS_t is the change in price, d_t is the change in time and dW_t is change in a Wiener process (brownian motion), μ is the mean and $f(t)$ is the volatility component. For simplicity, we make the following assumptions on $f(t)$:

(1) Volatility being constant between time 0 and t_1*
(2) Volatility depending on time during the period $t_1^* - t_2^*$ and then
(3) Volatility being constant in the time interval $\left(t_2^*, T\right)$ that is

$$f(t) = \begin{cases} \sigma, 0 < t \le t* \\ g(t), t_1^* < t \le t_2^*, \\ \sigma r, t_2^* < t \le T \end{cases}$$

where, $r \in R$; $r \ge 1$, $g(t)$ $r \in \mathbf{R}; r \ge 1$, $g(t)$ is assumed to be an exponential function. The time scale is defined by 0-t_1* as the time before speculation, $t_1^* - t_2^*$ the speculation period and t_2^* -T the time after speculation. We note that the speculation function *g (t)* can take on any other function.

Statistical Properties of the Improved OU Models

To solve for S_T, we recall (15) and expand to get:

$$dS_t + kS_t dt = k\mu dt + f(t)dW_t \tag{16}$$

If we multiply the integrating factor e^{kt} by (16), we get;

$$e^{kt} dSt + ke^{kt} Stdt = k\mu e^{kt} dt + f(t)e^{kt} dWt \tag{17}$$

It can be noted that the left side of (17) is the derivative of $e^{kt}S_t$ with respect to t. Thus,

$$^d\left(e^{kt} St\right) = k\mu e^{kt} dt + f(t)e^{kt} dWt \tag{18}$$

Integrating from 0 to T and evaluating the integrals, we get:

$$e^{kT}S_T - e^0 S_0 = k\mu \frac{e^{kT} - e^{k0}}{k} + \int_0^T f(t)e^{kt} dW_t \tag{19}$$

Multiplying by e^{-kT} to isolate S_T and moving S_0e^{-kT} to the right, we get the solution.

$$ST = S_0 e^{-kT} + \mu\left(1 - e^{-kT}\right) + e^{-kT}\int_0^T f(t)e^{kt} dW_t \tag{20}$$

0 Recalling the properties and assumptions of $f(t)$:

$$f(t) = \begin{cases} \sigma, 0 < t \le t* \\ g(t), t_1^* < t \le t_2^*, \\ \sigma r, t_2^* < t \le T \end{cases}$$

where, $r \in R$; $r \ge 1$ and $g(t)$ is assumed to be an exponential function such that $g(t) = \sigma e^{\gamma t}$ and $\gamma \ge 0$. Substituting $f(t)$ into (20), we get;

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$$S_T = S_0 e^{-kT} + \mu\left(1-e^{-kT}\right) + e^{-kT}\left[\int_0^{t_1^*} \sigma e^{kt}\, dW_t + \int_{t_1^*}^{t_2^*} g(t) ekt\, dW_t + \int_{t_2^*}^{T} r\sigma ekt\, dW_t \right] \quad (21)$$

Replacing $g(t)$ with $_\sigma e^{\gamma t}$ we have;

$$S_T = S_0 e^{-kT} + \mu\left(1-e^{-kT}\right) + e^{-kT}\left[\int_0^{t_1^*} \sigma e^{kt}\, dW_t + \int_{t_1^*}^{t_2^*} g(t) e^{kt}\, dW_t + \int_{t_2^*}^{T} r\sigma e^{kt}\, dW_t \right] \quad (22)$$

Expanding the last term of (22), we get;

$$\begin{aligned} S_T &= S_0 e^{-kT} + \mu\left(1-e^{-kT}\right) + e^{-kT} \\ &\int_0^{t_1^*} \sigma e^{-k} k\left(T-t\right) dW_t + \int_{t_1^*}^{t_2^*} \sigma e^{rt} e - k(T-t) dW_t + \int_{t_2^*}^{T} r\sigma e - k(T-t) dW_t \end{aligned} \quad (23)$$

S_T is normally distributed since the integral of the brownian motion part is Gaussian (Lebovits (2014), Shahnazi *et al*. (2021). To find the mean of S_T, we take the expectation on both sides of (23) and get;

$$\begin{aligned} E[S_T] &= E\left[S_0 e^{-kT} + \mu\left(1-e^{-kT}\right)\right] + \\ E&\left[\int_0^{t_1^*} \sigma e^{-k(T-t)} dW_t + \int_{t_1^*}^{t_2^*} \sigma e^{rt} e^{-k(T-t)}\, dW_t + \int_{t_2^*}^{T} r\sigma e - k(T-t) dW_t \right] \end{aligned} \quad (24)$$

Noting that the expected value of the brownian motion part is 0 and the expectation of a constant is the constant itself, we get;

$$E\left(S_T\right) = S_0 e^{-kT} + \mu\left(1 - e^{-kT}\right) \quad (25)$$

The variance of S_T is evaluated as follows;

$$\begin{aligned}
V(S_T) &= \left[E(S_T) - E(S_T)\right]^2 \\
&= E\left[\left(S0e - kT + \mu\right)\left(1 - e - kT\right) + \int_0^{t_1^*} \sigma e^{-k(T-t)} dW_t + \int_{t_1^*}^{t_2^*} \sigma e^{rt} e^{-k(T-t)}\, dW_t + \int_{t_2^*}^{T} r\sigma e^{-k(T-t)}\, dW_t - S_0 e^{-kT} + \mu\left(1-e^{-kT}\right) \right] \\
&= E\left[\int_0^{t_1^*} \sigma e^{-k(T-t)} dW_t + \int_{t_1^*}^{t_2^*} \sigma e^{rt} e^{-k(T-t)}\, dW_t + \int_{t_2^*}^{T} r\sigma e^{-k(T-t)}\, dW_t \right]^2 \\
&= E\left[\int_0^{t_1^*} \sigma e^{-2k(T-t)} (dWt)^2 + \int_{t_1^*}^{t_2^*} \sigma^2 e^{2rt} e^{-2k(T-t)} (dWt)^2 + \int_{t_2^*}^{T} r^2\sigma^2 e^{-2k(T-t)} (dWt)^2 \right] \\
&= E\left[\int_0^{t_1^*} \sigma e^{-2k(T-t)} dt + \int_{t_1^*}^{t_2^*} \sigma^2 e^{2rt} e^{-2k(T-t)}\, dt + \int_{t_2^*}^{T} r^2\sigma^2 e^{-2k(T-t)}\, dt \right] \\
&= \sigma^2 E\left[\int_0^{t_1^*} \sigma e^{-2k(T-t)} dt + \int_{t_1^*}^{t_2^*} \sigma^2 e^{2rt} e^{-2k(T-t)}\, dt + \int_{t_2^*}^{T} r^2\sigma^2 e^{-2k(T-t)}\, dt \right] \\
&= \sigma^2 E\left[\left(\frac{e^{-2k(T-t)}}{2k}\right)\Bigg| t^* + \left(\frac{e^{2rt-2k(T-t)}}{2r+2k}\right)\Bigg|_{t_1^*}^{t_2^*} + \left(\frac{r^{2e-2k(T-t)}}{2r+2k}\right)\Bigg|_{t^*2}^{T} \right] \\
&= \sigma^2 E\left[\frac{e^{-2k(T-t^*)} - e^{-2k(T-0)}}{2k} + \frac{e^{-2rt_2^*-2k(T-t_2^*)} - e^{2rt^*_1-2k}\left(T-t_1^*\right)}{2k} + \frac{r^2 e - 2k(T-T) - r^2 e^{-2k}(T-t_2^*)}{2k}\right] \\
&= \sigma^2 E\left[\left(\frac{e^{-2kT}.e^{2kt^*_1} - e^{-2kT} + r^2 e^{-2kT}.e^{2kT} - r^2 e^{-2kT}.e^{-2kT}.e^{2Kt^*_2}}{2k}\right) + \left(\frac{e^{2rt^*}2e-^{2kt^*}_2}{2r+2k}\right) - \frac{e^{2rt^*}1e - 2kT\, e^{2kt^*}_1}{2r+2k}\right] \\
&= \sigma^2 E\left[\frac{e^{-2}kT}{2k} E\left[\left(e^{2kt}*_1 - 1 + r^2\left(e^{2kT} - e^{2kt2^*}2\right)\right) + \left(\frac{e-^{2kT}}{2r+2k}\right)\left(e^{2t^*}_2(r+k) - e^{2t^*}_1(r+k)\right)\right]\right] \\
&= \sigma^2 E\left[\frac{e^{-2}kT}{2k}\left[\left(e^{2kt}*_1 - 1 + r^2\left(e^{2kT} - e^{2kt2^*}_2\right)\right)\right]\right] + \sigma^2\left[\left(\frac{e-^{2kT}}{2r+2k}\right)\left(e^{2t^*}_2(r+k) - e^{2t^*}_1(r+k)\right)\right] \quad (26) \\
&= \underbrace{\frac{\sigma^2 e^{-2kT}}{2k}\left[e^{2kt^*}_1 - 1\right]}_{term1} + \underbrace{\frac{\sigma^2 e^{-2kT}}{r+k}\left[e^{2kt^*}{}_2^{(r+k)}\right]}_{term2} + \underbrace{\frac{\sigma^2 e^{-2kT}}{2k}\left[r^2\left(e^{-2kT} - e^{2kt}*\right)\right]}_{term3} \\
&= \frac{1}{2}\left[\underbrace{\frac{\sigma^2 e^{-2kT}}{2k}\left[e^{2kt^*}_1 - 1\right]}_{term1} + \underbrace{\frac{\sigma^2 e^{-2kT}}{r+k}\left[e^{2t^*}{}_2^{(r+k)}\right]}_{term2} + \underbrace{\frac{\sigma^2 e^{-2kT}}{2k}\left[r^2\left(\left(e^{-2kT} - e^{2kt}*\right)\right)\right]}_{term3}\right]
\end{aligned}$$

Term 1 is the volatility before speculation, term 2 is the volatility during speculation and term 3 is the shift in volatility due to speculation occurring between t_2^* and T. In term 2, speculation is inversely proportional to γ+k. Noting that γ is the speculation parameter and the rate at which the exponential grows from t_1^* to t_2^*, a high value of γ indicates a great amount of speculation which causes volatility to shift rapidly and vice-versa. If $\gamma = 0$ (there is no speculation), volatility in term 3 is at a similar level as in term 1, with some adjustments of $t1*$ and $t2*$.

Numerical Simulations

Descriptive Analysis

To understand the movement of price over time, we plotted the average price for the period January 2019 to March 2021. The average price is the mean of the open, close, high and low daily prices. Figure 1 shows that between January 2019 and December 2019, the average price moved between 14 and 15 with an almost constant volatility.

From December 2019 there was a rise in price which occurred when Covid19 regulations such as international lockdowns were implemented. With the uncertantity of the future, market participants started to speculate causing the rise in price. This speculation lasted till around June 2020 when price began to drop. At this point, market participants got used to operating with the lockdown restrictions, noting how the market could continue running amid the pandemic. Therefore, price volatility began to stabilise from December 2020. Figure 1 provides clear evidence that speculation should be included in price prediction models such as the OU model, hence the importance of this study.

Numerical Simulations

Ornstein Uhlenbeck Models

Recalling (2) and generalizing S_T to an arbitrary start time t and end time t+δt we get;

$$S_t +_{St} = S_{te}^{-k\delta t} + \mu(1\text{-e}^{-k\delta t}) + \sigma\int_t^{t+st} e - ^{kst} e^{-k\delta t} dW_t \tag{27}$$

Therefore, the mean and variance of (27) becomes;

$$E(S_t +_{\delta t}) = S_t \,\mathrm{e} - k^{\delta t} + \mu(1 - \mathrm{e}^{-k\delta t}) \tag{28}$$

$$V\left(S_t +_{\delta t}\right) = \frac{\sigma^2}{2K}\left(1 - \mathrm{e}^{-k\delta t}\right) \tag{29}$$

The process in (27) at time t+δt from starting time t is normally distributed with the mean and variance given in (28 and (29), respectively. Therefore, $S_{t+\delta t}$ can be simulated using (30) since the integral of the brownian part in (27) becomes the standard normal times the standard error of the process.

$$S_t + \delta t = S_t e^{-kst} + \sigma\sqrt{\frac{1-e^{-2kst}}{2k}} N[0,] \tag{30}$$

Since (30) is modelled per year and we assumed there are 252 trading days in a year, we interpret δt as the small change in time such that $\delta t = (\frac{1}{252})$. The process presented in (30) is similar to the Autoregressive process of order 1 (AR(1)), in time series analysis. If we write the AR(1) process in terms of y;

$$y_{i+}1 = by_i + a + \epsilon i_+ 1 \tag{31}$$

where ϵ represents the errors which are assumed to be normally distributed (i.e SE). Mapping (31) to (30), we get the following parameters

$$\begin{aligned} & y_i + 1 = S_t +_{St} \\ & y_i = St \\ & b = e^{-kst} \\ & \alpha + \mu\left(1 - e^{-kst}\right) \\ & SE + \sigma\sqrt{\frac{1-e^{-kst}}{2k}} \end{aligned} \tag{32}$$

Considering (30), (31) and (32), we fit the $\mathrm{AR}(1)$ process to the data and use (32) to get the parameters of our stochastic process. The estimated parameters of the AR (1) process for the above stated data are; b = 0.999770392, a = 0.003705613, SE = 0.091078178. Using these values, we calculate the values of the parameters of the stochastics process (that is k, μ and σ) and get; k = 0.057867938, μ = 16.13884515 and σ = 0.091078178.

Improved OU Model

Similarly, recalling (23) and generalizing S_T to an arbitrary start time t and end time t+δt we get;

$$\begin{aligned} & S_t +_{St} = S_t e^{-kst} + \mu\left(1 - e^{-st}\right) + \\ & \int_0^{t_1^*} \sigma e^{-kst} dW_t \int_{t_1^*}^{t_2^*} \sigma e^{rt} e^{-kst} dW_t + \int_{t_2^*}^{t+st} r\sigma e^{-kst} dW_t \end{aligned} \tag{33}$$

Therefore, the mean and variance of (33) becomes;

$$E(S_t +_{\delta t}) = S_t \mathrm{e}^{-k\delta t} + \mu(1 - \mathrm{e}^{-k\delta t}) \tag{34}$$

$$\begin{aligned} & V(S_t +_{St}) = \sigma^2\left(\frac{e^{-2}kT}{2k}\left[e^{2kt\,{}^*_1} - 1\right] + \frac{e^{-2}kT}{2k} r^2\left(e^{2k\,(t+st)} - e^{2kt^*}{}_2\right)\right. \\ & \left. + \frac{e^{-2}kst}{2r+2k}\left[e^{-2rt_2^*(r+k)} e^2 t^*{}_1 (r+k)\right]\right) \end{aligned} \tag{35}$$

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The process in (33) normally distributed with the mean and variance given in (34) and (35) respectively. Therefore, $S_{t+\delta t}$ can be simulated using (36) since the integral of the Brownian parts in (33) becomes the standard normal multiplied by the standard error of the process.

$$S_t +_{\delta t} = S_t \mathrm{e}^{-k\delta t} + \mu(1 - \mathrm{e}^{-\delta t}) + \left(\sqrt{\sigma^2\left[\frac{e^{-2k(T-t)}}{2k}\left[e^{-2kt_1^*}-1\right]+\frac{e^{-2kst}}{2k}\left[r^2\left(e^{2k(t+st)}\right)-e^{-2kt_2^*}\right] + \frac{e^{-2kst}}{2r+2k}\left[e^{2t_2^*(r+k)}e^{2t_1^*(r+k)}\right]\right]}\right)N[0.1] \quad (36)$$

The process in (36) is fitted to the $(\mathrm{AR}(1))$ producing the same values for parameters b = 0.999770392, a = 0.003705613, k = 0.057867938 and μ =16.13884515. If we let $\lambda = 0.2, t_1^* = \frac{100}{252}, t_2^* = \frac{120}{252}$ and r=1.3 we get; SE=0.091078178 and by substituting the above values into:

$$\sigma \mathrm{Im}\, provedOU = \frac{SE}{\sqrt{\frac{e^{-2kst}}{2k}\left[e^{-2kt_1^*}-1\right]+\frac{e^{-2kst}}{2k}\left[r^2\left(e^{2k(t+st)}-e^{2kt_2^*}\right)\right]+\frac{e^{-2kst}}{2r+2k}\left[e^{2kt_2^*(r+k)}e^{2t_1^*(r+k)}\right]}}$$

$, we\, ob-tain; \sigma \mathrm{Im}\, provedOU = 0.07514189335$

Sample paths of S_T

Using the parameters (k, μ, $\sigma_{ImprovedOU}$, γ, r, t, t_1^*, t_2^*, T, SE) obtained in section 3.2.1 and 3.2.2 we generated S_t shown in Fig. 2 with S_0 = 15.4375 at time 0, forecasted over a time period of 252 days which exclude all non trading days such as weekends and public holidays. The stochasticity of S_t is seen in Fig. 2.

Distribution of S_T Over Different Lead Times

If we simulate S_T at different lead time intervals of δ_t such that $\delta t = \frac{63}{252}$ represents price distribution at 3 months (red line), $\delta t = \frac{126}{252}$ represents price distribution at 6 months (blue line), $\delta t = \frac{183}{252}$ represents price distribution at 9 months (green line) and $\delta t = \frac{252}{252}$ represents price distribution at 12 months (black line), we get the distribution in Fig. 3. The distribution of S_T for the OU model in (a), illustrates simulating that S_T over a smaller interval of δt predicts a value for S_T that has smaller variation compared to simulating S_T with a larger interval of δt. The distribution of S_T for the Improved OU model in (b), illustrates that the Improved model does not limit predictions of S_T to any specific time, as variance from the mean at different lead times is almost equal. Therefore, the Improved model indicates that variance becomes consistent w.r.t time and efficient since variance is low.

Expectation and Variance of S_T

Figure 4 displays the simulations of the expected mean and variance of S_T for the OU and Improved OU models, displayed as OU expectation (a), OU variance (b), Improved OU expectation (c) and Improved OU variance(d).

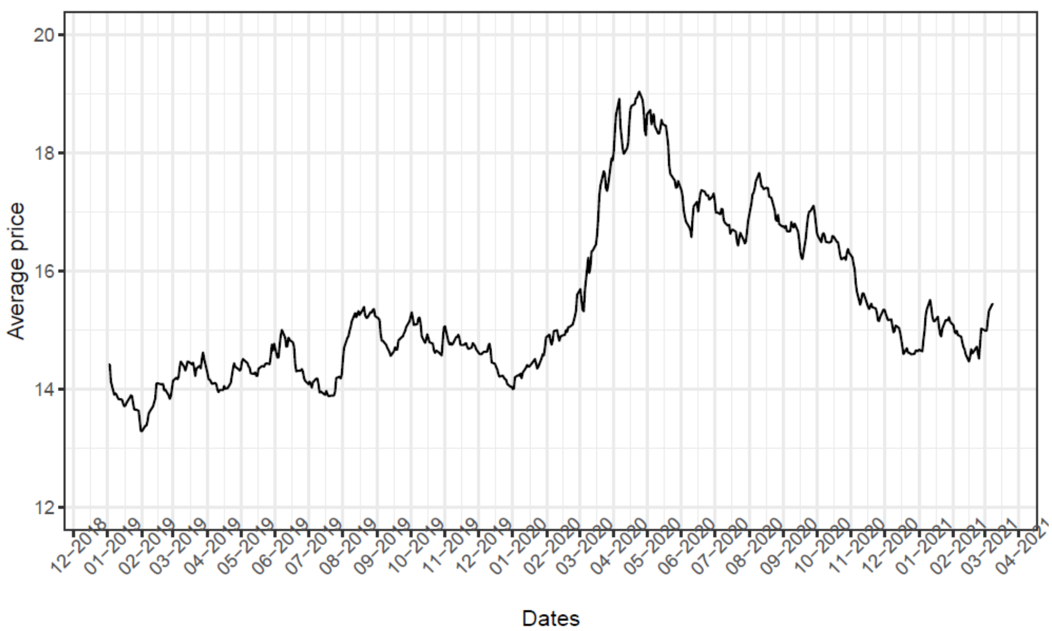

Fig. 1: Time series plot of average price for the time period January 2019 - March 2021

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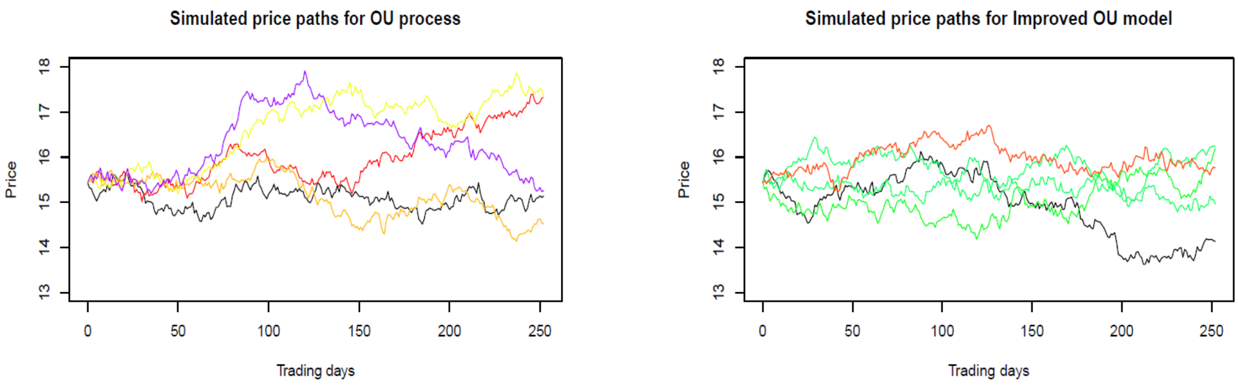
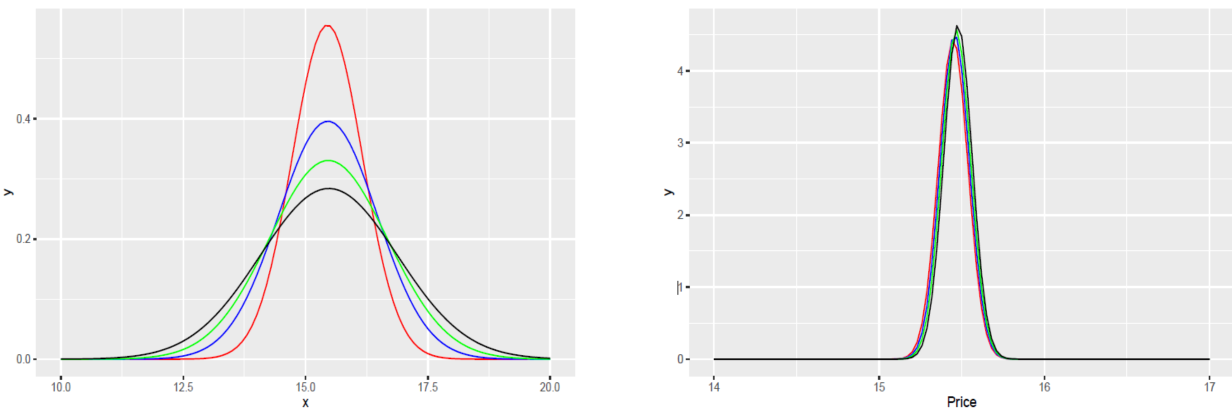


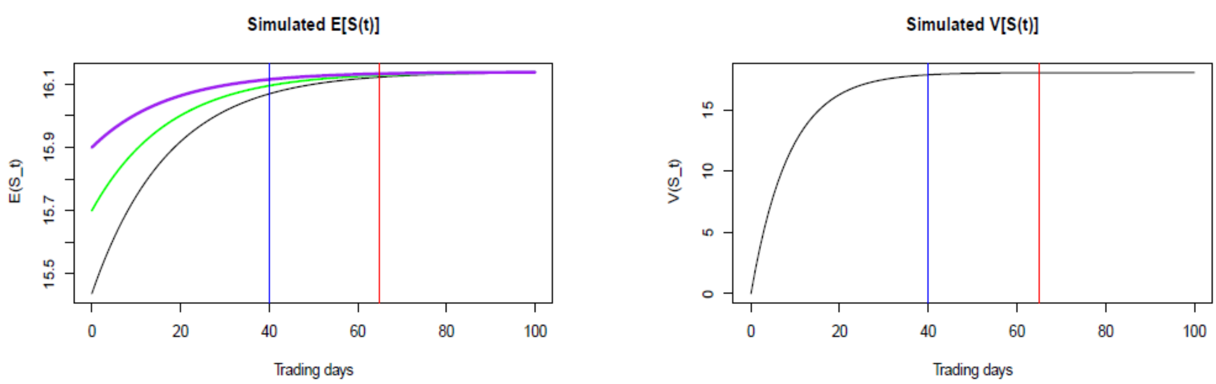
(a) (b)

Fig. 2: Price simulation of the OU and Improved OU models



(a) (b)

Fig. 3: Price distributions for the OU and Improved OU models



(a) (b)

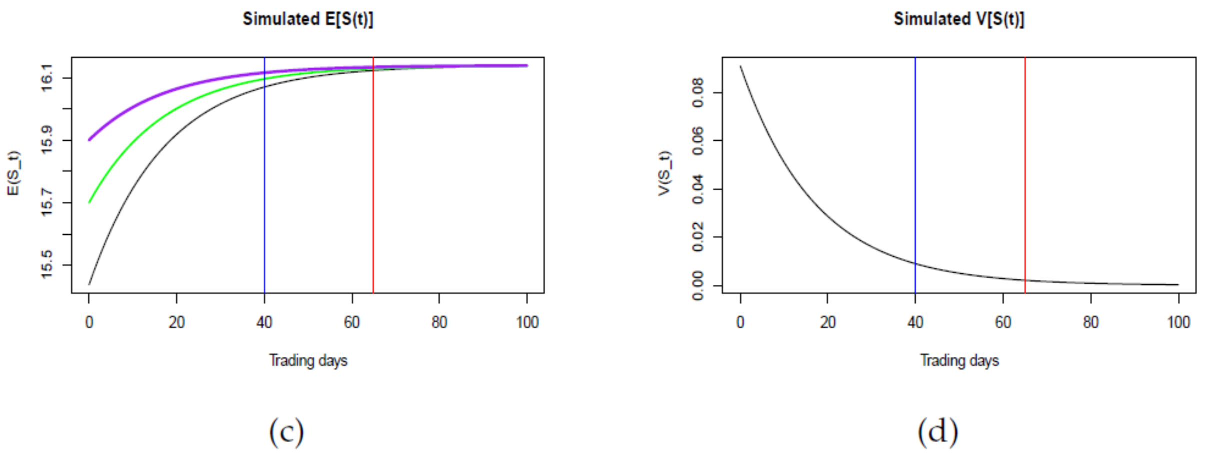


Fig. 4: Simulation of the expected OU mean (a), expected OU variance (b), expected Improved OU mean (c) and expected Improved OU variance (d)

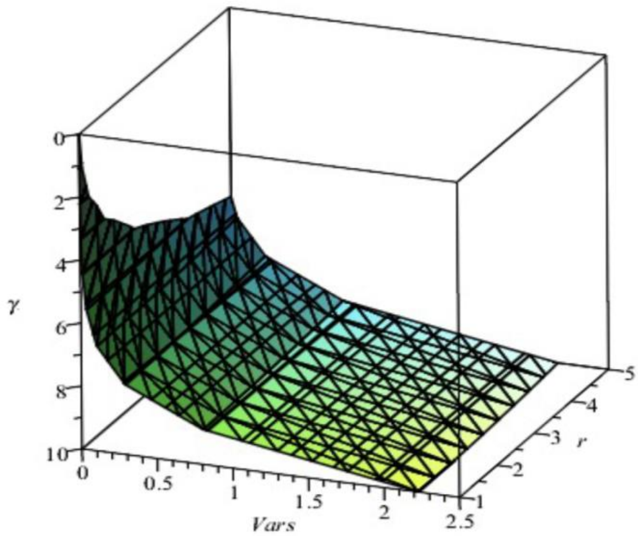


Fig. 5: Effect of speculation parameters (γ and r) on variance

The simulated expectations of S_T for both models in (a) and (c) behave exactly the same regardless of the starting value for S_T and stabalised upwards and became constant after some time. However, the simulated variances for the models as seen in (b) and (d) behaved differently from each other. The variance for the OU model (b), stabalised upwards around day 40 with high values for variance whereas, the simulated variance for the Improved OU model (d) stabalised downwards before day 40 with very low values for variance. This suggests that the OU model can forecast an accurate value for S_T with low volatility for short lead time and as the lead time increases, volatility becomes constant regardless of the starting price. Furthermore, it proves that the Improved model can forecast an accurate value for S_T with very low volatility regardless of the starting price.

We observe not only a decrease in volatility when speculation is included in the forecasting model but, we also observe that the variance itself is much lower in the Improved model compared to that of the OU model. Variance becomes constant in both models after some time. Therefore, what distinguishes the two models is the ability of the Improved model to produce constant and low

variance with the inclusion of market speculation. The speculation parameter is discussed and analysed in the next section.

The Effects of the Speculation Parameter (γ and r) on Variance

Using (36) and the parameters obtained in 3.2.1 and 3.2.2, we evaluate the effects of the speculation parameters (γ and r) on variance, as shown in Fig. 5.

It can bee seen that an increase in γ causes an increase in variance. However, as γ continues to increase, variance starts to stabilise at some point. Futhermore, noting that variance between t_2^* and T is rσ, we see that r, which is the shift in variance due to speculation, increases as γ increases. However, r does not have a direct effect on variance as it only effects variance through γ such that, the effect of r on volatility is not felt when γ becomes zero. Therefore, volatility is experienced when there is some value of γ. The presence of speculation in the market causes an increase in variance and the incorporation of this parameter in forecasting gives us a realistic model.

Conclusion

The OU model was modified by replacing its volatility component with an exponential function to cater for speculation which occurs in almost every reallife market. This model was named the Improved OU model as shown in (23) and was simulated along with the OU model using the same data described in section 1. It was found that the OU model can only predict price (S_T) with small volatility over a short period of time, whereas the Improved OU model can predict S_T with small volatility at any point in time during a trading year. The expectation of S_T was exactly the same in both the OU and the Improved OU models. However, it was found that variance of the OU model gradually increased until it reached its maximum and stabalised upwards after some time during the trading year, at an exchange rate of 16, whereas variance of the Improved OU model gradually decreased towards zero which proves that the incorporation of speculation in modelling reduces volatility. Although the OU model's volatility also stabilises towards zero, the Improved model stabilised at a faster rate than the OU model.

This study provides market participants with a forecasting model that takes speculation into consideration by introducing an exponential function as the speculation function. Future studies may use other types of functions and compare them against the current Improved OU model. Additionally, this study assumed an additive speculation model. We recommend to further study other models such as multiplicative. Future studies can also look at alternative forcasting approaches such as Recurrent Neural Network to forecast time series data other than the mentioned stochastic modelling approaches.

Author's Contributions

All authors equally contributed in this work.

Ethics

This article is original and contains unpublished material. The corresponding author confirms that all of the other authors have read and approved the manuscript and no ethical issues involved.

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