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Theory of Exact Trigonometric Periodic Solutions to Quadratic Liénard Type Equations

Jean Akande, Damien Kolawolé Kêgnidé Adjaï and Marc Delphin Monsia

Department of Physics, University of Abomey-Calavi, Abomey-Calavi, 01. BP.526, Cotonou, Benin

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Corresponding Author:
Jean Akande
Department of Physics,
University of Abomey-Calavi,
Abomey-Calavi, 01. BP.526,
Cotonou, Benin
Email: Jeanakande7@gmail.com

Abstract: A mathematical theory is developed through generalized Sundman transformation to show the existence of classes of quadratic Liénard type equations which admit exact and explicit general trigonometric solutions but with amplitude-dependent frequency. The application of the theory to compute also exact and explicit general periodic solutions to nonlinear differential equations like inverted Painlevé-Gambier equations in terms of trigonometric or Jacobian elliptic functions is highlighted by some illustrative examples.

Keywords: Liénard Equations, Painlevé-Gambier Equations, Periodic Solution, Generalized Sundman Transformation

Introduction

Consider the quadratic Liénard type nonlinear differential equation:

$$\ddot{x} + u(x)\dot{x}^2 + v(x) = 0 \qquad (1)$$

where, $u(x)$ and $v(x)$ are arbitrary functions of x and a dot over a symbol denotes a differentiation with respect to time. Mathews and Lakshmanan (1974) showed that Equation (1) admits a trigonometric solution with amplitude-dependent frequency if:

$$u(x) = \mp\frac{\lambda x}{1 \pm \lambda x^2}, \quad and \quad v(x) = \frac{\omega^2 x}{1 \pm \lambda x^2}$$

where, λ and ω are arbitrary constants. The present work is an extension of the result of Mathews-Lakshmanan (1974) by showing that there are other classes of functions $u(x)$ and $v(x)$ such that Equation (1) is closely connected to the linear harmonic oscillator equation and may exhibit exact and explicit general trigonometric solutions with amplitude-dependent frequency. In other words, the present research contribution shows the existence of classes of quadratic Liénard type equations which admit exact and explicit general trigonometric solutions but with amplitude-dependent frequency. This is accomplished by nonlocal transformation of the linear harmonic oscillator equation by means of generalized Sundman transformation. In this way the fundamental question to be solved first is the formulation of the appropriate generalized Sundman transformation (section 2) which may allow one to map secondly the linear harmonic oscillator equation into a general class of mixed Liénard type equations (section 3) from which it become possible to deduce the class of equation allowing one to find the quadratic Mathews-Lakshmanan equation (section 4) and another class of quadratic Liénard type equations exhibiting exact trigonometric solution but with amplitude dependent frequency (section 5). Finally the ability of the theory to be applied in finding exact and explicit general periodic solutions to nonlinear differential equations in terms of trigonometric or Jacobian elliptic functions is highlighted by illustrative examples (section 6) and concluding remarks are addressed for the work.

Generalized Sundman Transformation

In this section the generalized Sundman transformation required to demonstrate the preceding prediction is formulated. Such a transformation is a nonlocal transformation which applies to map in general a second order nonlinear ordinary differential equation into a second order linear ordinary differential equation (Monsia *et al*., 2016a) to find exact closed-form solutions. However, given a general second order linear ordinary differential equation, the generalized Sundman transformation, conversely, may be applied for detecting general classes of second order nonlinear ordinary differential equation exactly integrable (Monsia *et al*., 2016a). This formalism has been applied in (Monsia *et al*., 2016a; 2016b) to deduce mainly a class of quadratic Liénard type differential equations which admit exact trigonometric function solutions. Consider now the general second order linear differential equation of the form:

$$y'' + by' + a^2 y = 0 \tag{2}$$

where, prime means differentiation with respect to τ, a and b are arbitrary parameters and the generalized Sundman transformation (Monsia *et al.*, 2016a):

$$\begin{aligned} &y(\tau) = F(t,x),\ \ d\tau = G(t,x)dt, \\ &G(t,x)\frac{\partial F(t,x)}{\partial x} \neq 0 \end{aligned} \tag{3}$$

such that:

$$F(t,x) = \int g(x)^l, G(t,x) = \exp(\gamma\varphi(x)).$$

l and γ are arbitrary parameters, $g(x) \neq 0$ and $\varphi(x)$ are arbitrary functions of x. So, the application of Equation (3) to Equation (2) may give the desired general class of mixed Liénard type differential equations (Monsia *et al.*, 2016a).

General Class of Mixed Liénard Type Equations

By application of the generalized Sundman transformation (3), the general second order linear ordinary differential Equation (2) may be mapped onto a general class of mixed Liénard type equations (Monsia *et al.*, 2016a). Thus consider the following theorem.

Theorem 1

Consider Equation (2). Then by application of generalized Sundman transformation (3), Equation (2) reduces to:

$$\begin{aligned} &\ddot{x} + \left(l\frac{g'(x)}{g(x)} - \gamma\varphi'(x)\right)\dot{x}^2 + b\dot{x}\exp(\gamma\varphi(x)) \\ &+a^2\frac{\exp(2\gamma\varphi(x))\int g(x)^l dx}{g(x)^l} = 0 \end{aligned} \tag{4}$$

where, prime denotes the differentiation with respect to x.

Proof

The use of Equation (3) allows one to compute the first derivative of $y(\tau)$:

$$y'(\tau) = \dot{x}g(x)^l e^{-\gamma\varphi(x)} \tag{5}$$

from which, the second derivative:

$$y''(x) = \left[\ddot{x} + \left(l\frac{g'(x)}{g(x)} - \gamma\varphi'(x)\right)\dot{x}^2\right]g(x)^l e^{-2\gamma\varphi(x)} \tag{6}$$

Substituting Equations (5) and (6) into Equation (2), knowing that $y(\tau) = \int g(x)^l dx$ leads, after a few mathematical manipulations, to obtain Equation (4).

On the other hand, by application of $l = \gamma$ and $\varphi(x) = \ln(g(x))$, Equation (4) reduces to:

$$\ddot{x} + b\dot{x}g(x)^l + a^2 g(x)^l \int g(x)^l dx = 0 \tag{7}$$

where, ln designates the natural logarithm.

For $l = 1$, $b = 1$ and $a^2 = \frac{2}{9}$, Equation (7) reduces to the Musielak equation (Musielak, 2008).

The choice $\varphi(x) = \ln(f(x))$ gives as equation:

$$\begin{aligned} &\ddot{x} + \left(l\frac{g'(x)}{g(x)} - \gamma\frac{f'(x)}{f(x)}\right)\dot{x}^2 + b\dot{x}f(x)^\gamma \\ &+a^2\frac{f(x)^{2\gamma}\int g(x)^l dx}{g(x)^l} = 0 \end{aligned} \tag{9}$$

For $l = \gamma = 1$, $g(x) = f(x) = U'(x)$, $b = \frac{3}{2}$ and $a^2 = \frac{1}{2}$, Equation (9) becomes the equation studied in (Cariñena *et al.*, 2005):

$$\ddot{x} + \frac{3}{2}\dot{x}U'(x) + \frac{1}{2}U'(x)U(x) = 0 \tag{10}$$

where, $U(x)$ is an arbitrary function such that by making $U(x) = k\,x^2$, Equation (10) reduces to the well known modified Emden equation:

$$\ddot{x} + 3k\,x\dot{x} + k^2x^3 = 0 \tag{11}$$

which is widely studied in the literature. However this equation may directly be obtained from Equation (9) by putting $l = \gamma = \frac{1}{2}$, $g(x) = f(x) = x^2$, $b = 3k$ and $a = k$.

Now, from Equation (4), one may deduce the general class of quadratic Liénard type differential equations from which the class of quadratic Liénard type equations which admit trigonometric solutions (Monsia *et al.*, 2016a; 2016b) may be obtained.

General Class of Quadratic Liénard Type Equations

Consider now the following useful theorem.

Theorem 2

Let $b = 0$. *Then Equation (4) reduces to:*

$$\begin{aligned} &\ddot{x} + \left[l\frac{g'(x)}{g(x)} - \gamma\varphi'(x)\right]\dot{x}^2 \\ &+\frac{a^2e^{2\gamma\varphi(x)}\int g(x)^l dx}{g(x)^l} = 0 \end{aligned} \tag{12}$$

Proof

The theorem 2 is a special case ($b = 0$) of theorem 1.

The Equation (12) represents the required general class of quadratic Liénard type differential equations. One may notice that a judicious parametric choice as well as an appropriate selection of function $g(x)$ and $\varphi(x)$ may lead to interesting nonlinear oscillator equations for mathematical physics. An interesting case of Equation (12) may be, for $\varphi(x) = \ln(f(x))$, obtained as:

$$\ddot{x}+\left[l\frac{g'(x)}{g(x)}-\gamma\frac{f'(x)}{f(x)}\right]\dot{x}^2 +\frac{a^2 f(x)^{2\gamma}\int g(x)^l dx}{g(x)^l}=0 \tag{13}$$

So with that some examples related to Equation (13) may be given to illustrate moreover the high mathematical significance of the work developed in this study.

Examples

Illustrative examples related to Equation (13) are given in this subsection.

The parametric choice $l=\frac{1}{2}$ and $\gamma = 1$, in Equation (13) yields:

$$\ddot{x}+\frac{1}{2}\left[\frac{g'(x)}{g(x)}-2\frac{f'(x)}{f(x)}\right]\dot{x}^2 +\frac{a^2 f(x)^2 (g(x))^{\frac{1}{2}}\int (g(x))^{\frac{1}{2}} dx}{g(x)}=0 \tag{14}$$

This case corresponds to the class of quadratic Liénard type differential equations constructed in (Mustafa, 2015). Indeed, the substitution of $l=\frac{1}{2}$, $\gamma = 1$ and $\varphi(x) = \ln(f(x))$, into the nonlocal transformation (3), yields the generalized Sundman transformation introduced in (Mustafa, 2015) to build the generalized position-dependent mass Euler-Lagrange equation (Equation (12) of (Mustafa, 2015)) identical to Equation (14). The Equation (14) is shown in (Mustafa, 2015) to include as special cases several interesting position-dependent mass nonlinear oscillator equations like the celebrated Mathews-Lakshmanan equations, the quadratic Morse type equation, etc. It is therefore no longer necessary to again carry out these calculations to prove the usefulness of the generalized Sundman transformation (3) introduced in this study. Let us consider, nevertheless, other interesting illustrative examples in addition to those mentioned in (Mustafa, 2015) for $\gamma \neq 0$ and $l \neq 0$.

Making now $f(x) = x^2$ and $g(x) = x$, Equation (13) becomes:

$$\ddot{x}+(l-2\gamma)\frac{\dot{x}^2}{x}+\frac{a^2}{l+1}x^{4\gamma+1}=0 \tag{15}$$

The use of nonlocal transformation (3) leads to:

$$y(\tau)=\frac{1}{l+1}x^{l+1}, d\tau = x^{2\gamma}\, dt,\ l\neq -1 \tag{16}$$

that is:

$$x(t)=\left[(l+1)A_0\right]^{\frac{l}{l+1}}\sin^{\frac{1}{l+1}}\left(a\phi(t)+\alpha\right) \tag{17}$$

where, α is an arbitrary constant and the function $\tau = \phi(t)$ satisfies:

$$\left[(l+1)A_0\right]^{\frac{2\gamma}{l+1}}(t+K)=\int\frac{d\phi(t)}{\sin^{\frac{2\gamma}{l+1}}\left(a\phi(t)+\alpha\right)} \tag{18}$$

with K an integration constant and:

$$y(\tau)=A_0\sin\left(a\tau+\alpha\right) \tag{19}$$

is the solution to Equation (2) where $b = 0$.

Now let $\gamma \neq 0$ and $l=-\frac{2}{3}$. Then Equation (15) reduces to:

$$\ddot{x}-\left(\frac{2}{3}+2\gamma\right)\frac{\dot{x}^2}{x}+3a^2x^{4\gamma+1}=0 \tag{20}$$

such that Equation (18) becomes:

$$\left(\frac{A_0}{3}\right)^{6\gamma}(t+K)=\int\frac{d\phi(t)}{\sin^{6\gamma}\left(a\phi(t)+\alpha\right)} \tag{21}$$

In this perspective the solution $x(t)$ reads:

$$x(t)=\frac{A_0^3}{9}\sin^3\left[a\phi(t)+\alpha\right] \tag{22}$$

where, $\phi(t)$ satisfies Equation (21). For an integer γ or $\gamma=\frac{1}{6}$, Equation (21) may be easily computed (Gradshteyn and Ryzhik, 2007) and the solution $x(t)$ may be expressed explicitly in terms of elementary functions. Conversely, for a non-integer γ, Equation (21) will be evaluated in terms of special functions and the argument

$a\phi(t) + \alpha$, becomes a complicated function of t. So, for example, $\gamma = \frac{1}{6}$, Equation (21) gives:

$$\frac{A_0}{3}(t+K) = \frac{1}{a}\ln\left(tg\frac{a\phi(t)+\alpha}{2}\right) \quad (23)$$

that is (Gradshteyn and Ryzhik, 2007):

$$\exp\left(\frac{aA_0}{3}(t+K)\right) = tg\frac{a\phi(t)+\alpha}{2} \quad (24)$$

and the solution $x(t)$ may be written as:

$$x(t) = \frac{A_0^3}{9}\sin^3\left[2tg^{-1}\left(\exp\left(\frac{aA_0}{3}(t+K)\right)\right)\right] \quad (25)$$

Now, the parametric choice $\gamma = -\frac{3}{2}$ and l = -2, according to Equation (15), leads to the equation:

$$\ddot{x} + \frac{\dot{x}^2}{x} - \frac{a^2}{x^5} = 0 \quad (26)$$

Following Equation (16) the solution $x(t)$ takes the form:

$$x(t) = -\frac{1}{A_0\sin\left(a\phi(t)+\alpha\right)} \quad (27)$$

where, $\phi(t)$ satisfies:

$$-A_0^3(t+K) = \int\frac{d\phi(t)}{\sin^3\left(a\phi(t)+\alpha\right)} \quad (28)$$

that is (Gradshteyn and Ryzhik, 2007):

$$-\frac{\cos\left(a\phi(t)+\alpha\right)}{2a\sin^2\left(a\phi(t)+\alpha\right)} + \frac{1}{2a}\ln\left[tg\left(\frac{a\phi(t)+\alpha}{2}\right)\right] = -A_0^3(t+K) \quad (29)$$

The Equation (26) admits a position-dependent mass dynamics so that the mass $m(x) = m_0x^2$ and the potential energy function $V(x) = \frac{m_0a^2}{2}\frac{1}{x^2}$, where m_0 is an arbitrary constant. Such a potential is the so-called singular inverse square potential and has been widely studied in the quantum mechanics. However it is for the first time the existence of such a potential has been mathematically established through a nonlinear oscillator equation. This highlights the physical importance of Equation (26). Let us consider now other interesting classes of quadratic Liénard type differential equations obtained for $l = 0$, or $\gamma = 0$.

Let $\gamma = 0$. In this situation Equation (12) reduces to:

$$\ddot{x} + l\frac{g'(x)}{g(x)}\dot{x}^2 + \frac{a^2\int\left(g(x)\right)^l dx}{\left(g(x)\right)^l} = 0 \quad (30)$$

By application of $g(x) = h'(x)$, where $h(x)$ is an arbitrary function of x, Equation (30) becomes:

$$\ddot{x} + l\frac{h''(x)}{h'(x)}\dot{x}^2 + \frac{a^2\int\left(h'(x)\right)^l}{\left(h'(x)\right)^l} = 0 \quad (31)$$

So, for $l = 1$, Equation (31) gives:

$$\ddot{x} + \frac{h''(x)}{h'(x)}\dot{x}^2 + \frac{\omega_0^2h(x)}{h'(x)} = 0 \quad (32)$$

where, $a^2 = \omega_0^2$. In Tiwari *et al*. (2013) Equation (32) has been classicaly analyzed whereas in (Gubbiotti and Nucci, 2014), Equation (32) was studied from quantum viewpoint. On the other hand it is worth to mention that the class of quadratic Liénard type Equation (15) has been recently shown to be able to exhibit exact trigonometric solutions of harmonic form but with amplitude-dependent frequency (Nonti *et al*., 2018). Now it may be possible to show the existence of another class of quadratic Liénard type equations which has the ability to exibit trigonometric solutions of harmonic form.

Another Class of Quadratic Liénard Type Equations Which Admit Exact Trigonometric Solutions

Let us take now into account the case $l = 0$, which appears to be of high interest since it highlights the trigonometric function solutions to a class of quadratic Liénard type equations. For $l = 0$, or $g(x) = 1$, Equation (12) reduces to (Monsia *et al*., 2016b):

$$\ddot{x} - \gamma\varphi'(x)\dot{x}^2 + a^2xe^{2\gamma\varphi(x)} = 0 \quad (33)$$

According to the generalized Sundman transformation (3) and Equation (19) the solution to Equation (33) may read:

$$x(t) = A_0\sin\left(a\phi(t)+\alpha\right) \quad (34)$$

where, $\tau = \phi(t)$ obeys:

$$d\phi(t) = \exp\left(\gamma\varphi(x)\right)dt \quad (35)$$

that is:

$$\exp\left(-\gamma\varphi(x)\right)d\phi(t) = dt \quad (36)$$

The above shows clearly the following result.

Theorem 3

If l = 0, or g(x) = 1, Equation (12) reduces to Equation (33), then Equation (34) becomes the solution to Equation (33) where, $\phi(t)$ satisfies Equation (36).

So the problem of finding $x(t)$ reduces to solve Equation (36) once the function $\varphi(x)$ and the parameter γ are defined. The comparison of equation (33) with Equation (1) specifies the new class of functions $u(x) = -\gamma\varphi'(x)$ and $v(x) = a^2xe^{2\gamma\varphi(x)}$, for which Equation (1) **may admit** exact trigonometric solutions with amplitude-dependent frequency. That being so, some illustrative examples are studied in this paragraph. An interesting case is to consider $\varphi(x) = \ln(f(x))$ such that Equation (33) becomes:

$$\ddot{x} - \gamma\frac{f'(x)}{f(x)}\dot{x}^2 + a^2x\left(f(x)\right)^{2\gamma} = 0 \quad (37)$$

and (36) yields:

$$dt = \frac{d\phi(t)}{\left(f(x)\right)^{\gamma}} \quad (38)$$

Let $\gamma = 2$ and $\varphi(x) = \frac{1}{2}\ln(1+\mu x)$. Then Equation (37) takes the form:

$$\ddot{x} - \frac{\mu}{1+\mu x}\dot{x}^2 + a^2x\left(1+\mu x\right)^2 = 0 \quad (39)$$

The Equation (38) gives in this perspective:

$$(t+K) = \int\frac{d\phi(t)}{1+\mu A_0\sin\left(a\phi(t)+\alpha\right)} \quad (40)$$

that is (Gradshteyn and Ryzhik, 2007):

$$a\phi(t)+\alpha = 2tg^{-1}\left[\sqrt{1-\mu^2A_0^2}\,tg\left(\frac{a(t+K)\sqrt{1-\mu^2A_0^2}}{2} - \mu A_0\right)\right] \quad (41)$$

so that the solution:

$$x(t) = A_0 \times\sin\left[2tg^{-1}\left[\sqrt{1-\mu^2A_0^2}\,tg\left(\frac{a(t+K)\sqrt{1-\mu^2A_0^2}}{2}\right) - \mu A_0\right]\right] \quad (42)$$

An appropriate choice of initial conditions, that is of constants A_0 and K may lead Equation (42) to exhibit harmonic behavior. In this perspective Fig. 1 shows the harmonic form of Equation (42) in solid line compared with the solution obtained by numerical integration of Equation (39) in circles line under the conditions that $A_0 = 0{,}01$; $\mu = 0{,}25$; $a = 1$; $x_0 = 0{,}01$ and $\dot{x}_0 = 0$.

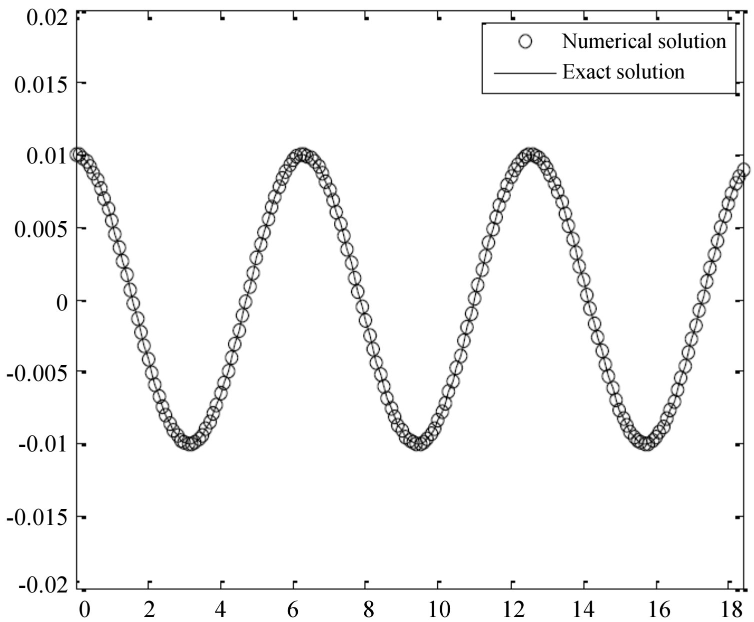

Fig. 1: Comparison of solution (42) with numerical solution of Equation (39). Typical representation values are: $A_0 = 0{,}01$; $\mu = 0{,}25$; $a = 1$; $x_0 = 0{,}01$ and $\dot{x}_0 = 0$

By applying γ = -1 and $f(x)=\frac{\sqrt{1-\mu^2x^2}}{\sqrt{1+\mu^2(1-\mu^2x^2)}}$, Equation (37) gives:

$$\ddot{x}-\frac{\mu^2x}{(1-\mu^2x^2)\left[1+\mu^2(1-\mu^2x^2)\right]}\dot{x}^2+\frac{a^2x\left[1+\mu^2(1-\mu^2x^2)\right]}{1-\mu^2x^2}=0 \tag{43}$$

The Equation (43) admits then as solution:

$$x(t)=B_0\cos(a\phi(t)+\beta) \tag{44}$$

where, B_0 and β are arbitrary parameters, such that:

$$a(t+K)=\int\frac{\sin\theta}{\sqrt{1+\mu^2\sin^2\theta}}d\theta \tag{45}$$

where, $\mu^2B_0^2=1$ and θ = $a\phi(t)+\beta$.

Therefore one may obtain:

$$\cos(a\phi(t)+\beta)=\frac{\sqrt{\mu^2+1}}{\mu}\sin\left[-\mu a(t+K)\right] \tag{46}$$

so that Equation (44) takes the expression:

$$x(t)=\frac{\sqrt{\mu^2+1}}{\mu^2}\sin\left[-\mu a(t+K)\right] \tag{47}$$

By application of suitable initial conditions, one may consider:

$$K=-\frac{\pi}{2\mu a} \tag{48}$$

such that the definitive solution:

$$x(t)=\frac{\sqrt{\mu^2+1}}{\mu^2}\sin\left(\mu at+\frac{\pi}{2}\right) \tag{49}$$

Figure 2 shows the harmonic behavior of Equation (49) in solid line compared with the solution obtained by numerical integration of Equation (43) in circles line under the conditions:

$$\mu=6,5; a=0,5; x_0=0,15 \ and \ \dot{x}_0=0.$$

On the other hand, it is worth to note that for $\gamma=1$ and the function $f(x)=\frac{1}{\sqrt{1+\mu x^2}}$, Equation (37) reduces to the equation of motion of a particle moving on a rotating parabola:

$$\ddot{x}+\frac{\mu x}{1+\mu x^2}\dot{x}^2+\frac{a^2x}{1+\mu x^2}=0 \tag{50}$$

which **may admit** according to the present theory an exact trigonometric solution but with amplitude-dependent frequency. However the detailed study of this equation will be performed in a subsequent work. It is now interesting to show that the proposed theory of nonlinear differential equations may be also used to solve exactly some inverted Painlevé-Gambier equations (Ince, 1956).

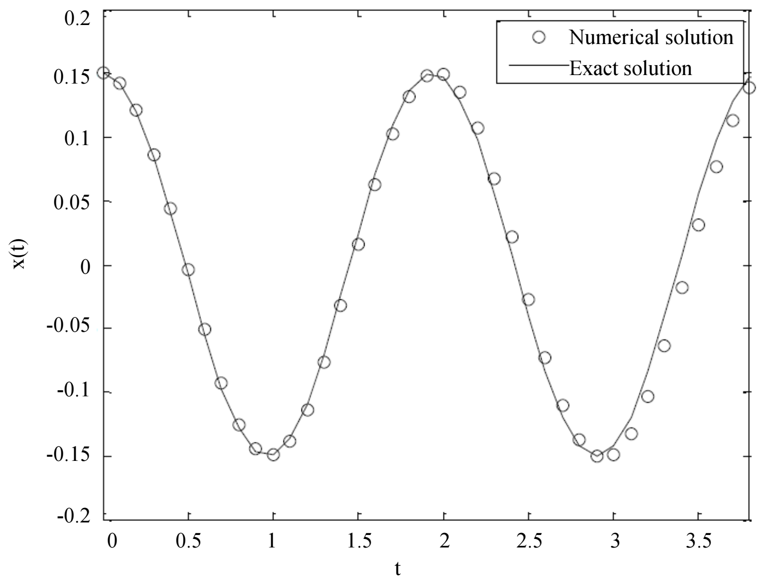

Fig. 2: Comparison of solution (49) with numerical solution of Equation (43). Typical representation values are: μ = 6,5; a = 0,5; x_0 = 0,15 and $\dot{x}_0=0$

Inverted Painlevé-Gambier Equations

In this paragraph, solutions to some inverted Painlevé-Gambier equations are expressed as mentioned in the above.

Inverted Painlevé-Gambier XVIII Equation

The Painlevé-Gambier XVIII equation may read (Ince, 1956):

$$\ddot{x}-\frac{1}{2}\frac{\dot{x}^2}{x}-4x^2=0 \tag{51}$$

so its inverted version becomes:

$$\ddot{x}=\frac{1}{2}\frac{\dot{x}^2}{x}+4x^2=0 \tag{52}$$

which only differs from Equation (51) by a sign. The Equation (52) belongs to the class of quadratic Liénard type nonlinear differential equation represented by Equation (33) under the considerations that $\gamma=\frac{1}{4}$, $\varphi(x) = \ln(x^2)$ and $a^2 = 4$. As a result the solution to Equation (52) may be expressed following Equation (34) as:

$$x(t)=B_0\cos\left(a\phi(t)-\beta\right) \tag{53}$$

such that $\phi(t)$ satisfies:

$$dt=B_0^{-\frac{1}{2}}\cos^{-\frac{1}{2}}\left(a\phi(t)-\beta\right)d\phi(t) \tag{54}$$

The integration of the right hand side of Equation (54) may be evaluated as:

$$J=B_0^{-\frac{1}{2}}\int\frac{d\phi(t)}{\sqrt{\cos(a)\phi(t)-\beta}} \tag{55}$$

that is (Gradshteyn and Ryzhik, 2007):

$$J=\frac{2}{a\sqrt{B_0}}\int\frac{d\psi}{\sqrt{-1+2\cos^2\psi}} \tag{56}$$

where:

$$a\phi=2\psi+\beta \tag{57}$$

According to (Gradshteyn and Ryzhik, 2007):

$$J=\frac{\sqrt{2}}{a}B_0^{-\frac{1}{2}}F\left(\delta,\frac{\sqrt{2}}{2}\right) \tag{58}$$

where, $\delta=\arcsin\left(\sqrt{2}\sin\psi\right)$ and $F(\theta, k)$, is the elliptic integral of the first kind. So, one may find $a\phi(t)$-β, such that ψ is given by:

$$\cos\left[\sin^{-1}\left(\sqrt{2}\sin\psi\right)\right]=cn\left(\frac{a\sqrt{2B_0}}{2}t,\frac{\sqrt{2}}{2}\right) \tag{59}$$

Therefore, one may recover the explicit solution (53) under the definitive form:

$$x(t)=B_0\cos\left[2\sin^{-1}\left[\frac{\sqrt{2}}{2}\sin\left[\cos^{-1}\left[cn\left(\frac{a\sqrt{2B_0}}{2}t,\frac{\sqrt{2}}{2}\right)\right]\right]\right]\right] \tag{60}$$

where, $a = 2$ and $cn(z, k)$ is the Jacobi elliptic function. Now one may reduce Equation (60) using appropriate trigonometric identities (Gradshteyn and Ryzhik, 2007) to:

$$x(t)=B_0cn^2\left(\sqrt{2B_0}t,\frac{\sqrt{2}}{2}\right) \tag{61}$$

in order to show that Equation (60) may exhibit periodic solution behavior. The solution (60) allows in principle, to compute the exact solution of the initial Painlevé-Gambier XVIII equation by replacing the parameter a by ia, where i is the purely imaginary number. In other words, the solution (60) gives the exact solution to the Painlevé-Gambier XVIII Equation (51) by replacing $cn\left(\sqrt{2B_0}t,\frac{\sqrt{2}}{2}\right)$ by (Gradshteyn and Ryzyhik, 2007):

$$cn\left(i\sqrt{2B_0}t,\frac{\sqrt{2}}{2}\right)=\frac{1}{cn\left(\sqrt{2B_0}t,\frac{\sqrt{2}}{2}\right)} \tag{62}$$

Inverted Painlevé-Gambier XXXII Equation

The Painlevé-Gambier XXXII equation (Ince, 1956) is written as:

$$\ddot{x}-\frac{1}{2}\frac{\dot{x}^2}{x}+\frac{1}{2x}=0 \tag{63}$$

The inverted version may then be written in the form:

$$\ddot{x}-\frac{1}{2}\frac{\dot{x}^2}{x}-\frac{2a^2}{x}=0 \tag{64}$$

where, $a^2=\frac{1}{4}$. The Equation (64) may be obtained by substituting $\gamma=-\frac{1}{2}$ and $l=-\frac{3}{2}$ in Equation (15). In this perspective the solution takes the form following (17):

$$x(t) = \frac{4}{A_0^2 \sin^2(a\phi(t) + \alpha)} \tag{65}$$

where, $\phi(t)$ obeys:

$$\frac{A_0^2}{4}(t + K) = \int \frac{d\phi(t)}{\sin^2(a\phi(t) + \alpha)} \tag{66}$$

or:

$$\frac{A_0^2}{4}(t + K) = -\frac{1}{a}\cot(a\phi(t) + \alpha) \tag{67}$$

Therefore the preceding solution x(t) becomes:

$$x(t) = \frac{4}{A_0^2 \sin^2\left[\cot^{-1}\left(-\frac{aA_0^2}{4}t + \frac{aA_0^2}{4}K\right)\right]} \tag{68}$$

Inverted Painlevé-Gambier XXI Equation

According to (Ince, 1956) the Painlevé-Gambier XXI equation reads:

$$\ddot{x} - \frac{3}{4}\frac{\dot{x}^2}{x} + 1 = 0 \tag{69}$$

In this regard, the inverted equation may be written as:

$$\ddot{x} - \frac{3}{4}\frac{\dot{x}^2}{x} - 1 = 0 \tag{70}$$

which may be obtained from Equation (15) by setting $\gamma = -\frac{1}{4}$, $l = -\frac{5}{4}$ and $a^2 = \frac{1}{4}$. Thus the solution $x(t)$ becomes:

$$x(t) = \frac{256}{A_0^4 \sin^4(a\phi(t) + \alpha)} \tag{71}$$

where, $\phi(t)$ satisfies:

$$\frac{A_0^2}{16}(t + K) = \int \frac{d\phi(t)}{\sin^2(a\phi(t) + \alpha)} \tag{72}$$

So, Equation (71) may be written in the form:

$$x(t) = \frac{256}{A_0^4 \sin^4\left[\cot^{-1}\left(-\frac{aA_0^2}{16}(t + K)\right)\right]} \tag{73}$$

Now, taking into account these illustrative examples a conclusion may be addressed for the work.

Concluding Remarks

If the problem of determining approximate trigonometric periodic solutions to nonlinear differential equations has been more or less solved, the problem of finding exact and analytical general trigonometric solutions to nonlinear differential equations is yet an active mathematical research field. In such a situation the Liénard type nonlinear equations are subject of intensive study from mathematical viewpoint as well as physical standpoint. Different linearizing transformations with different complexities have been used to construct exact periodic solutions to Liénard nonlinear differential equations. In particular the generalized Sundman transformation has been widely used to establish exact solutions to diverse types of Liénard differential equations. Conversely such a transformation may also be used to detect diverse classes of Liénard differential equations having exact analytical solutions. In this perspective, a generalized Sundman transformation is introduced in this study to highlight a general class of quadratic Liénard type nonlinear differential equations which admit exact and explicit general trigonometric solutions with amplitude-dependent frequency. By doing so, it appears that the proposed nonlinear differential equation theory may be used to exactly solve a number of interesting mixed and quadratic Liénard type equations as well as to generate new generalized nonlinear differential equations of Liénard type for mathematical modeling and creation of new dynamical systems characterized in particular by a harmonic potential and a position-dependent mass.

Author's Contributions

All authors contributed to the development and formulation of this work.

Ethics

The authors declare that there exists no competing interests. All authors read and approved the final manuscript for publication.

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