

# The Linear and Quadratic Nature of Sequences for Determining Prime Numbers by Elimination of Composites

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**Abstract:** A previously derived set of sequences for generating prime numbers is modified by various algebraic and trigonometric manipulations. The modified sequences are then used to more efficiently generate prime numbers. The modified sequences are simplified from their original form and are much easier to implement into a computer program. It is shown by computation that the modified sequences more efficiently determine prime numbers because the modified sequences provide flexibility with their implementation. The added flexibility allows for the removal of redundancy in the overall process, which decreases the amount of data to process; thus, the efficiency of the prime number generator is significantly improved.

**Keywords:** Primes, Sequences, Linear, Quadratic, Equations

## Introduction

In “Sequences for Determination of Prime Numbers by Elimination of Composites,” it was proven that there exists a function, simply termed the  $(a)$  function, that will generate all prime numbers starting with 5 up to some number  $(N)$ , and the function generates composites periodically (Dugas and O'Connor, 2017). The  $(a)$  function is as shown in Equation 1:

$$a(i) = 6i - a(i-1) \quad (1)$$

$$1 \leq i \leq N$$

It was also shown that the  $(a)$  function can be written as shown in Equation 2:

$$a(i) = 1 + \sum_{\ell=1}^i \left( 4 \left| \sin \left( \frac{\pi \ell}{2} \right) \right| + 2 \left| \cos \left( \frac{\pi \ell}{2} \right) \right| \right) \quad (2)$$

$$1 \leq i \leq N$$

There are many alternative ways to represent such an equation. For example and as shown in Equation 3, the squares of sine and cosine can be used in place of the absolute values of sine and cosine, respectively:

$$a(i) = 1 + \sum_{\ell=1}^i \left( 4 \sin^2 \left( \frac{\pi \ell}{2} \right) + 2 \cos^2 \left( \frac{\pi \ell}{2} \right) \right) \quad (3)$$

Using the half angle formulas for the sine squared and cosine squared terms, Equation 4 is obtained:

$$a(i) = 1 + \sum_{\ell=1}^i \left( 4 \left( \frac{1 - \cos(\pi \ell)}{2} \right) + 2 \left( \frac{1 + \cos(\pi \ell)}{2} \right) \right) \quad (4)$$

Equation 4 can then be simplified as shown in Equation 5 (Mandal and Asif, 2007):

$$a(i) = 1 + \sum_{\ell=1}^i (3 - \cos(\pi \ell)) \quad (5)$$

$$1 \leq i \leq N$$

In this study, the  $(a)$  function will be modified such that the summation operator is removed. The summation operators for the  $(i_{(not\ prime)})$  equation, which is the equation used to determine the composites generated by the  $(a)$  function, are removed as well. By removing these summation operators, computer resources will be preserved because more flexibility is established in choosing which values of  $(i_{(not\ prime)})$  need to be calculated and which values can be ignored. The following equation was used in “Sequences for Determination of Prime Numbers by Elimination of Composites” to determine the indices of  $(i)$  that correspond to composites generated by the  $(a)$  function (Dugas and O'Connor, 2017):

$$\left\{ \begin{aligned}
 i_{(not\ prime)} &= D + \sum_{j=2}^m \left[ p \left| \sin\left(\frac{\pi j}{2}\right) + q \left| \cos\left(\frac{\pi j}{2}\right) \right| \right] u(n+1-j) \\
 D &= \sum_{k=1}^n \left( 4k \left| \cos\left(\frac{\pi k}{2}\right) \right| + 8k \left| \sin\left(\frac{\pi k}{2}\right) \right| \right) \\
 p &= 7 + \sum_{k=2}^n \left( 2 \left| \cos\left(\frac{\pi k}{2}\right) \right| + 6 \left| \sin\left(\frac{\pi k}{2}\right) \right| \right) u(n-2) \\
 q &= 2n+1 \\
 2 &\leq j \leq m \\
 1 &\leq k \leq n \\
 1 &\leq i_{(not\ prime)} \leq N
 \end{aligned} \right\} \quad (6)$$

The derivation of Equation 6 will not be explained; however, it will be shown that the components comprising Equation 6 can be alternatively represented by linear and quadratic equations, which significantly simplifies its form (Dugas and O'Connor, 2017). The modified (*a*) function and the simplified form of Equation 6 will then be used to generate prime numbers. From a computer science point of view, the efficiency of the overall prime number generator can be improved by simplifying the (*a*) function and Equation 6. The primary reason the efficiency will be improved is the removal of the summation operators from the (*a*) function and Equation 6. By removing the summation operators from Equation 6, it becomes possible to determine which of its values need to be calculated and which are redundant and can be negated. It becomes possible to negate redundancy in Equation 6 by the removal of the summation operators because the current values are no longer dependent on any previous values of Equation 6. The same is true for the (*a*) function. If it is determined that an index of the (*a*) function will generate a composite number, that index can be omitted because the (*a*) function is no longer dependent on any of its previous values. The purpose of this work is to present a modified form of the (*a*) function and Equation 6, report on the linear and quadratic nature of Equation 6, and present a set of equations that could potentially be used to more efficiently generate prime numbers. The validity of the method will be proven by computation using computer software to determine all prime numbers between one and one billion.

### Experimental Procedure

Equation 7 is a mathematical representation of the algorithm used to calculate all prime numbers between one and one billion, using the C++ programming language. The algorithm in Equation 7 is a significantly simplified version of the prime number algorithm used in previous experiments (Dugas and O'Connor, 2017). The derivation and specifics of Equation 7 will be explained in detail in the Discussion section. However, it is worth noting that Equation 7 is a more efficient prime number generator because there are no summation operators:

$$P(n,m) = \left\{ \begin{aligned}
 a(n) &= 3n + 1 + \sin^2\left(\frac{\pi n}{2}\right) \\
 i_{(not\ prime)} &= 3nm + m + m \sin^2\left(\frac{\pi n}{2}\right) + n + n \sin^2\left(\frac{\pi n}{2}\right) \\
 &\quad + \sin^2\left(\frac{\pi m}{2}\right) \sin^2\left(\frac{\pi n}{2}\right) \\
 1 &\leq m \\
 1 &\leq n \leq N \\
 1 &\leq i_{(not\ prime)} \leq N \\
 D &= (3n^2 + 2n) + (2n+1) \sin^2\left(\frac{\pi n}{2}\right) \\
 n &\neq D \\
 n &\neq i_{(not\ prime)}, \text{ for } 1 \leq n \leq 7 \text{ and all } (m)
 \end{aligned} \right\} \quad (7)$$

The prime number generators used in previous experiments possessed a total of four summation operators (Dugas and O'Connor, 2017). By removing the summation operators, the number of calculations the program performs can be reduced because more flexibility in deciding which values of (*i*<sub>(not prime)</sub>) are needed and which values can be negated is established. Equation 7 shows that (*n*) does not need to equal any values generated by the equation for (*D*) or any value of (*i*<sub>(not prime)</sub>) for ( $1 \leq n \leq 7$ ) and all (*m*), which eliminates a significant portion of the redundancy in (*i*<sub>(not prime)</sub>). The specifics of why (*n*) will generate redundancy for those values of (*D*) and (*i*<sub>(not prime)</sub>) will also be explained in the Discussion section.

The flow diagram in Fig. 1 shows the logic used to calculate the values of the (*a*) function and (*i*<sub>(not prime)</sub>) and determine prime numbers (Gaddis and Krupnpw, 2007). The values of the (*i*<sub>(not prime)</sub>) equation were calculated first for all (*n*) and (*m*), such that ( $1 \leq i_{(not\ prime)} \leq N$ ). A nested For loop was used to calculate all values of (*i*<sub>(not prime)</sub>), as shown in Fig. 1. The outer For loop pertains to the (*n*) variable and the inner For loop pertains to the (*m*) variable. If (*n*) equaled any value of (*i*<sub>(not prime)</sub>) for ( $1 \leq n \leq 7$ ) and all (*m*), that value of (*n*) was negated and the inner loop pertaining to (*m*) was also negated. It should be noted that (*m*) is chosen such that (*i*<sub>(not prime)</sub>) never exceeds (*N*), as indicated in Equation 7 where the limits on (*i*<sub>(not prime)</sub>) are ( $1 \leq i_{(not\ prime)} \leq N$ ) (Gaddis and Krupnpw, 2007).

Once all needed values of (*i*<sub>(not prime)</sub>) were calculated, they were arranged into 17 different segments, which were arranged in sequential order. Each segment was stored in one of 17 different text files. The next step was to begin calculating the values of the (*a*) function for all values of (*n*) not equal to any value of (*i*<sub>(not prime)</sub>). The values of (*i*<sub>(not prime)</sub>) were read from the text files in which they were stored and if (*n*) was not equal to any of the values of (*i*<sub>(not prime)</sub>), the value of the (*a*) function was calculated and output to the prime number text file. The text file was to contain all primes between one and one billion after the program executed.

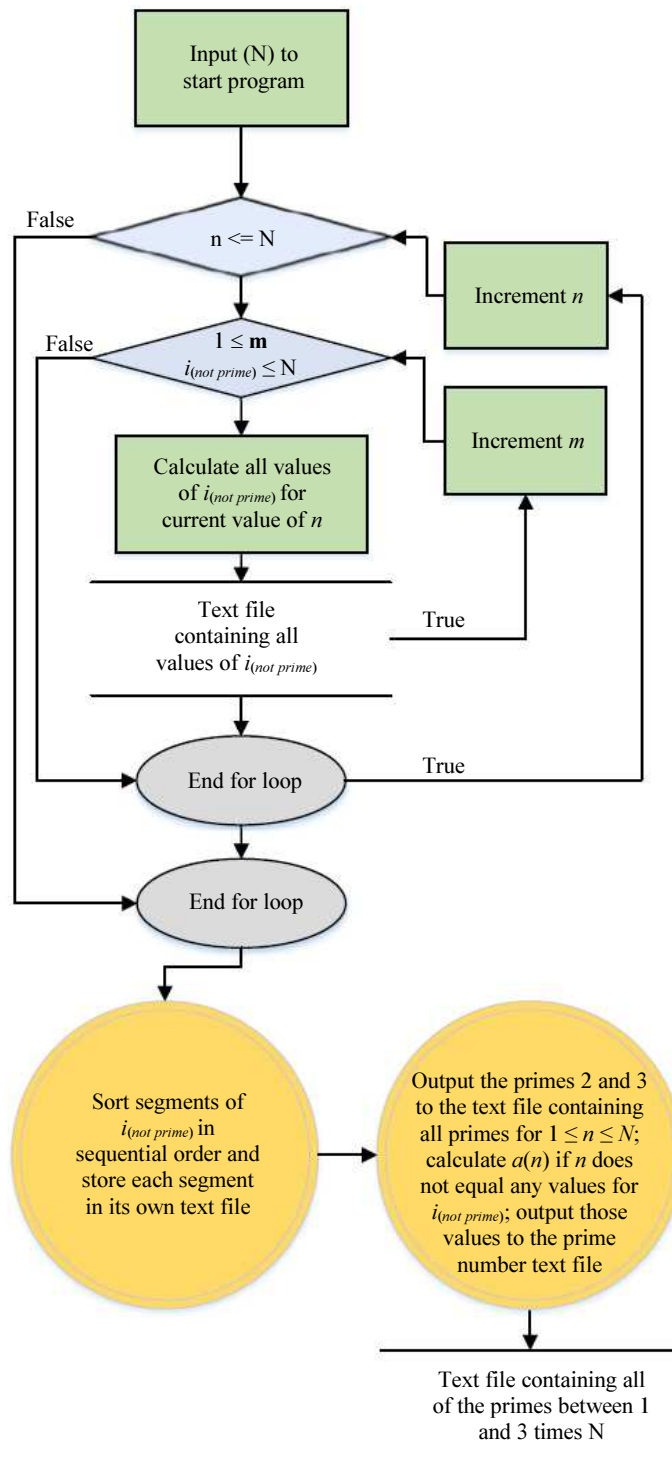


Fig. 1: Flow diagram of prime number generator

**Results**

After the described program executed, there were 50,847,534 integers in the prime number text file. There are exactly that many prime numbers between one and

one billion. The last prime number in the text file was 999,999,937, which is the last prime number between one and one billion. Table 1 contains the density of primes for all primes in the text file for  $(3N)$  approximately equal to powers of ten up to one billion.

**Table 1:** Prime number density results for prime number generator for  $10^x \cong 3 \times N, 1 \leq x \leq 9$

Approximately 3 times N	n + 2	a(n)
10	4	7
100	25	97
1,000	168	997
10,000	1,229	9,973
100,000	9,592	99,991
1,000,000	78,498	999,983
10,000,000	664,579	9,999,991
100,000,000	5,761,455	99,999,989
1,000,000,000	50,847,534	999,999,937

The results are in good agreement with the known density of primes data for powers of ten between one and one billion and the prime number theorem, which states that the density of prime numbers should decrease as (N) approaches infinity (Crandall and Pomerance, 2006). All integers in the prime number text file were in fact prime numbers, the density of primes decreased for larger values of (N) and the experiment successfully validated the reasoning used to simplify the prime number generator.

## Discussion

### The Linear Nature of the (a) Function

To modify the (a) function such that the summation operator is removed, begin with Equation 8 (Mandal and Asif, 2007):

$$a(n) = 1 + \sum_{\ell=1}^n \left( 4 \left( \frac{1 - (-1)^\ell}{2} \right) + 2 \left( \frac{1 + (-1)^\ell}{2} \right) \right) \quad (8)$$

$1 \leq n \leq N$

Equation 8 simplifies to Equation 9, which is very similar to Equation 5:

$$a(n) = 1 + \sum_{\ell=1}^n (3 - (-1)^\ell) \quad (9)$$

$1 \leq n \leq N$

The only difference in Equation 5 and 9 is that the cosine term in Equation 5 has been replaced with a (-1) raised to an appropriate exponent (Dugas and O'Connor, 2017). Equation 9 can be modified to obtain the two sums in Equation 10:

$$a(n) = 1 + \left( \sum_{\substack{\ell=1 \\ \ell \text{ odd}}}^n (4) + \sum_{\substack{\ell=2 \\ \ell \text{ even}}}^n (2) \right) \quad (10)$$

It can be directly shown that Equation 10 reduces to the two lines in Equation 11:

$$a(n) = \begin{cases} 3n + 2, & \text{for } n \text{ odd} \\ 3n + 1, & \text{for } n \text{ even} \end{cases} \quad (11)$$

$1 \leq n \leq N$

If a single equation for the (a) function is desired, the (a) function can be written as shown in Equation 12:

$$a(n) = (3n + 2) \sin^2 \left( \frac{\pi n}{2} \right) + (3n + 1) \cos^2 \left( \frac{\pi n}{2} \right) \quad (12)$$

$1 \leq n \leq N$

Using the trigonometric identity in Equation (13):

$$\cos^2 \theta = 1 - \sin^2 \theta \quad (13)$$

Equation 12 reduces to Equation 14 (Mandal and Asif, 2007):

$$a(n) = 3n + 1 + \sin^2 \left( \frac{\pi n}{2} \right) \quad (14)$$

$1 \leq n \leq N$

The approach just used to simplify the (a) function is also used on more than one occasion to simplify ( $i_{(not\ prime)}$ ). Simplifying the (a) function and ( $i_{(not\ prime)}$ ) in this way allows for the summation operators in both equations to be eliminated, improving the overall efficiency of the prime number generator.

### The Linear and Quadratic Nature of ( $i_{(not\ prime)}$ )

To determine the indices of the composites generated by the (a) function so that the composites can be eliminated, Equation 6 or some form of Equation 6 should be used. Observe Fig. 2. The (D) component of Equation 6, which stands for the diagonal of ( $i_{(not\ prime)}$ ) when its values are inserted into a matrix of (k) rows and (j) columns, is shown in Equation 15, and the graph of (D) with respect to the indices of (k) is shown in Fig. 2 (Dugas and O'Connor, 2017):

$$Diagonal = \sum_{k=1}^n \left( 4k \left| \cos \left( \frac{\pi k}{2} \right) \right| + 8k \left| \sin \left( \frac{\pi k}{2} \right) \right| \right) \quad (15)$$

As shown in Fig. 2, (D) can be alternatively represented by two quadratic equations. The solid line, which has the appearance of a saw tooth plot, is a graph of all values generated by (D) for ( $1 \leq n \leq 7$ ). Figure 2 shows that (D) is comprised of two parabolas and both parabolas are plotted using dashed lines. The top parabola with respect to the saw-tooth line is utilized for odd values of (k) and the parabola on the bottom is utilized for even values of (k).

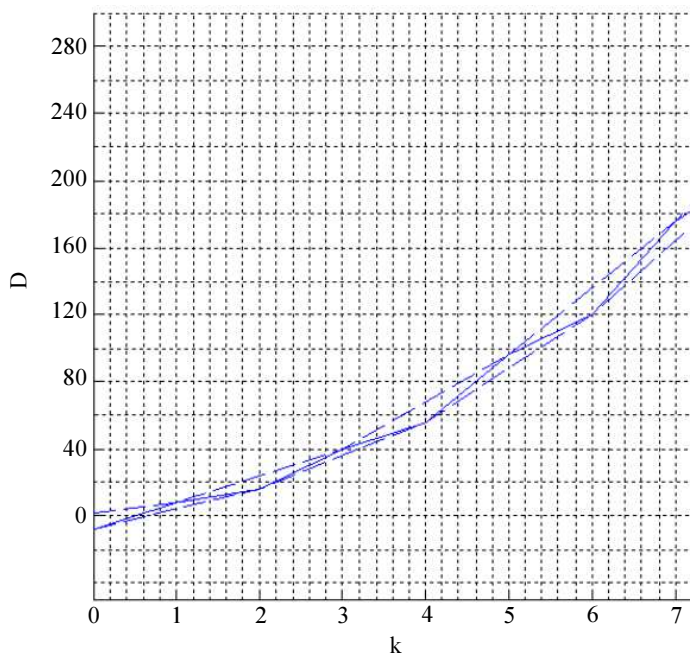


Fig. 2: Plot of the (D) diagonal of  $(i_{(not\ prime)})$  with respect to (k)

To derive the parabolas shown in Fig. 2, begin by substituting the absolute values of the cosine and sine terms in (D) with the trigonometric identities in Equation 16 and 17, respectively, to obtain Equation 18:

$$\left| \cos\left(\frac{\pi k}{2}\right) \right| = \frac{1 + (-1)^k}{2} = \begin{cases} 0, & \text{for } k \text{ odd} \\ 1, & \text{for } k \text{ even} \end{cases} \quad (16)$$

$$\left| \sin\left(\frac{\pi k}{2}\right) \right| = \frac{1 - (-1)^k}{2} = \begin{cases} 0, & \text{for } k \text{ even} \\ 1, & \text{for } k \text{ odd} \end{cases} \quad (17)$$

$$\begin{aligned} D &= \sum_{k=1}^n \left[ 4k \left( \frac{1 + (-1)^k}{2} \right) + 8k \left( \frac{1 - (-1)^k}{2} \right) \right] \\ &= \sum_{k=1}^n (6 - 2(-1)^k)k \\ &= \sum_{\substack{k=1 \\ k \text{ odd}}}^n (8k) + \sum_{\substack{k=1 \\ k \text{ even}}}^n (4k) \end{aligned} \quad (18)$$

If (n) is odd, it can be directly shown that

$$D = 3n^2 + 4n + 1. \quad (19)$$

Likewise if (n) is even, it can be directly shown that:

$$D = 3n^2 + 2n. \quad (20)$$

Equation 21 summarizes the two quadratics and there is a parabola for the odd values of (n) and a parabola for the even values of (n):

$$\begin{cases} 3n^2 + 4n + 1, & \text{for } n \text{ odd} \\ 3n^2 + 2n, & \text{for } n \text{ even} \end{cases} \quad (21)$$

If a single equation for (D) is desired, (D) can be written as shown in Equation 22:

$$D = (3n^2 + 4n + 1)\sin^2\left(\frac{\pi n}{2}\right) + (3n^2 + 2n)\cos^2\left(\frac{\pi n}{2}\right) \quad (22)$$

Using the trigonometric identity in Equation 13 yields Equation 23:

$$D = (3n^2 + 4n + 1)\sin^2\left(\frac{\pi n}{2}\right) + (3n^2 + 2n)\left(1 - \sin^2\left(\frac{\pi n}{2}\right)\right) \quad (23)$$

(D) can then be simplified as shown in Equation (24) (Mandal and Asif, 2007).

$$D = (3n^2 + 2n) + (2n + 1)\sin^2\left(\frac{\pi n}{2}\right) \quad (24)$$

(D) has now been written into quadratic components and simplified and the linear components of  $(i_{(not\ prime)})$  can be similarly derived. Observe Fig. 3. As (j) is iterated for (k = 1),  $(i_{(not\ prime)})$ , shown as a solid, blue, saw-tooth line, alternates between two lines that are shown as a pair of dashed, blue lines. There is a dashed line for the odd values of (j) and a dashed line for the even values of (j) and that is true for any given value of (k). As another example and for (k = 2),  $(i_{(not\ prime)})$  is shown as a solid, green, saw-tooth line alternating between two dashed,

green lines. For convenience, begin deriving these sets of lines by writing ( $i_{(not\ prime)}$ ) as shown in Equation 25 thru 27 (Dugas and O'Connor, 2017):

$$i_{(not\ prime)} = D + \sum_{j=2}^m \left( p \left| \sin\left(\frac{\pi j}{2}\right) \right| + q \left| \cos\left(\frac{\pi j}{2}\right) \right| \right) u(n+1-j) \quad (25)$$

$$p = 7 + \sum_{k=2}^n \left( 2 \left| \cos\left(\frac{\pi k}{2}\right) \right| + 6 \left| \sin\left(\frac{\pi k}{2}\right) \right| \right) u(n-2) \quad (26)$$

$$q = 2n + 1 \quad (27)$$

The ( $q$ ) component shown in Equation 27 is a line and no further derivations are needed. That is not the case with the ( $p$ ) component shown in Equation 26. To derive the lines to represent the ( $p$ ) component, begin by substituting the absolute values of the cosine and sine terms in ( $p$ ) with Equation 16 and 17, respectively, as shown in Equation 28:

$$p = 7 + \sum_{k=2}^n \left( 2 \left( \frac{1+(-1)^k}{2} \right) + 6 \left( \frac{1-(-1)^k}{2} \right) \right) u(n-2) \quad (28)$$

It can be directly shown that Equation 28 then reduces as shown in Equation 29:

$$p = 7 + \sum_{k=2}^n (4 - 2(-1)^k) u(n-2) \quad (29)$$

$$p = 7 + \left( \sum_{\substack{k=2 \\ k\ odd}}^n (6) + \sum_{\substack{k=2 \\ k\ even}}^n (2) \right) u(n-2) \quad (29)$$

Equation 29 can be written as Equation 30 for ( $n$ ) equivalent to an odd integer:

$$p = 7 + (4n - 2)u(n-2) \quad (30)$$

Likewise, Equation 29 can be written as Equation 31 for ( $n$ ) equivalent to an even integer:

$$p = 7 + (4n - 6)u(n-2) \quad (31)$$

The  $u(n-2)$  term in Equation 30 thru 31 can now be eliminated because if ( $n = 1$ ), an odd value, ( $p$ ) equals 7 (Zill and Cullen, 2008). The 7 term can be combined with the other terms in these equations, as summarized in Equation 32:

$$p = \begin{cases} 4n + 3, & \text{for } n \text{ odd} \\ 4n + 1, & \text{for } n \text{ even} \end{cases} \quad (32)$$

$1 \leq n \leq N$

If a single equation for ( $p$ ) is desired, ( $p$ ) can be written as shown in Equation 33:

$$p = (4n + 3)\sin^2\left(\frac{\pi n}{2}\right) + (4n + 1)\cos^2\left(\frac{\pi n}{2}\right) \quad (33)$$

Using the trigonometric identity in Equation 13, Equation 33 reduces to Equation 34 (Mandal and Asif, 2007):

$$p = 4n + 1 + 2\sin^2\left(\frac{\pi n}{2}\right) \quad (34)$$

$1 \leq n \leq N$

Using the simplifications for ( $D$ ) and ( $p$ ), Equation 6 can now be written as shown in Equation 35:

$$i_{(not\ prime)} = (3n^2 + 2n) + (2n + 1)\sin^2\left(\frac{\pi n}{2}\right) + \sum_{j=2}^m \left[ \left( 4n + 1 + 2\sin^2\left(\frac{\pi n}{2}\right) \right) \sin^2\left(\frac{\pi j}{2}\right) + (2n + 1)\cos^2\left(\frac{\pi j}{2}\right) \right] u(n+1-j) \quad (35)$$

Equation 35 can be simplified further by using the trigonometric identity in Equation 13 once more, which yields Equation 36 (Mandal and Asif, 2007):

$$i_{(not\ prime)} = (3n^2 + 2n) + (2n + 1)\sin^2\left(\frac{\pi n}{2}\right) + \sum_{j=2}^m \left[ 2n + 1 + 2 \left( n + \sin^2\left(\frac{\pi n}{2}\right) \right) \sin^2\left(\frac{\pi j}{2}\right) \right] u(n+1-j) \quad (36)$$

Equation 37, which is termed  $P(n, m)$ , is a simplified version of the prime number generator presented in previous research and it can be more efficiently implemented from a computer science point of view:

$$P(n, m) = \begin{cases} a_{prime}(n) = 3n + 1 + \sin^2\left(\frac{\pi n}{2}\right) \\ i_{(not\ prime)} = (3n^2 + 2n) + (2n + 1)\sin^2\left(\frac{\pi n}{2}\right) \\ + \sum_{j=n+1}^m \left[ 2n + 1 + 2 \left( n + \sin^2\left(\frac{\pi n}{2}\right) \right) \sin^2\left(\frac{\pi j}{2}\right) \right] u(n+1-j) \\ (n+1) \leq j \leq m \\ 1 \leq m \\ 1 \leq n \leq N \\ 1 \leq i_{(not\ prime)} \leq N \\ n \neq i_{(not\ prime)} \end{cases} \quad (37)$$

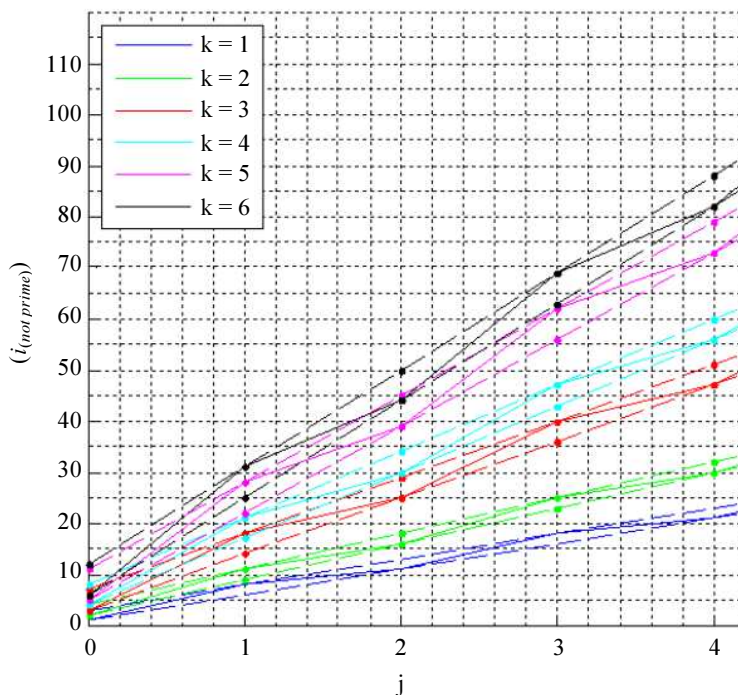


Fig. 3: Plot of  $(i_{(not\ prime)})$  with respect to  $(j)$  for different values of  $(k)$

In  $P(n, m)$ , the summations in  $(a)$ ,  $(D)$  and  $(p)$  have been removed by the previous simplifications. Those three components are no longer dependent on any previous values of  $(n)$ ; they are only dependent on the current value of  $(n)$ . Simplifying  $(i_{(not\ prime)})$  in this way allows for more flexibility in choosing which values of  $(n)$  are needed and which values of  $(n)$  cause  $(i_{(not\ prime)})$  to generate redundant values. In other words,  $(n)$  does not need to be equal to any of the values generated by  $(i_{(not\ prime)})$  because all of those values will be redundant. (Note that the limits of  $(j)$  and  $(m)$  are now modified such that  $(j)$  is never less than  $(n+1)$  and  $(m)$  is never less than 1. Setting the limits of  $(j)$  and  $(m)$  in this way better clarifies the range of values needed for  $(j)$  and  $(m)$ ).

Observe Table 2, which is a matrix containing the values of  $(i_{(not\ prime)})$  for  $(n)$  rows and  $(m)$  columns. Everything below the blue diagonal in red is redundant and the values on and above the diagonal are the only values needed, which is how the diagonal,  $(D)$ , was derived in previous research (Dugas and O'Connor, 2017). However, some values on and above the diagonal are redundant as well. A large portion of the remaining redundancy occurs whenever  $(n)$  equals any values generated by  $(i_{(not\ prime)})$ , which is indicated by the red rows on and above the blue diagonal in Table 2. If Table 2 is extended for higher values of  $(m)$ , the value of 208 in row 8 and the values of 408 and 431 in row 11 will also occur in row 1.

The remaining redundancy is slightly more complicated to identify and eliminate. Table 3 is an extension of Table 2 for higher values of  $(m)$  to provide more clarity. For  $(n)$  greater than one,  $(i_{(not\ prime)})$  equates to a redundant value if  $(m)$  equals any value of  $(i_{(not\ prime)})$  when  $(n)$  in  $(i_{(not\ prime)})$  is equal to one. Likewise for  $(n)$  greater than two,  $(i_{(not\ prime)})$  generates redundancy if  $(m)$  equals  $(i_{(not\ prime)})$  when  $(n)$  in  $(i_{(not\ prime)})$  is equal to two (Dugas and O'Connor, 2017).

For example, the value of 128 in column 18 also occurs in row 1. As another example, the value of 179 in column 16 also occurs in row 2. Now, a process for determining the remaining redundancy generated by the  $(i_{(not\ prime)})$  equation has been identified, however, the last summation operator needs to be eliminated so that the current values of  $(i_{(not\ prime)})$  are never dependent on any previous values of  $(i_{(not\ prime)})$ . The derivation used to eliminate the final summation operator from  $(i_{(not\ prime)})$  will not be explained, but using an approach that is similar to those used to simplify  $a(n)$ ,  $(D)$  and  $(p)$ , Equation (38) can be obtained for  $(i_{(not\ prime)})$ :

$$i_{(not\ prime)} = 3nm + m + m \sin^2\left(\frac{\pi n}{2}\right) + n + n \sin\left(\frac{\pi n}{2}\right) + \sin^2\left(\frac{\pi m}{2}\right) \sin^2\left(\frac{\pi n}{2}\right) \tag{38}$$

Equation 38 is a significantly simplified form of  $(i_{(not\ prime)})$ , it is much easier to use and it can be implemented much more efficiently into a computer program. Without the summation operators in  $(i_{(not\ prime)})$ , all of the values of  $(n)$  and  $(m)$  that cause  $(i_{(not\ prime)})$  to generate redundancy can be negated, which will make the entire process more efficient.

At this point, it is prudent to recognize a mathematical connection between the lines plotted in Fig. 3 and the  $(a)$  function, which provides a simpler alternative way to obtain Equation 38. Refer to Fig. 3.

The lines plotted in Fig. 3 have slopes dependent on the values of  $a(n)$  and intercepts dependent on the values of  $(n)$ . The system of lines are easy to derive graphically and are shown in Equation 39:

$$(i_{(not\ prime)}) = \begin{cases} ma(n) + n, & \text{for } n \text{ even and } n \text{ odd} \\ ma(m) + 2n + 1, & \text{for } m \text{ odd and } n \text{ odd} \\ ma(n) + n, & \text{for } m \text{ even and } n \text{ even} \\ ma(n) + 2n, & \text{for } m \text{ odd and } n \text{ even} \end{cases} \quad (39)$$

**Table 2:** Matrix containing values of  $(i_{(not\ prime)})$  with rows and  $(m)$  columns,  $1 \leq n \leq 13$ ,  $1 \leq m \leq 13$

m	1	2	3	4	5	6	7	8	9	10	11	12	13
n = 1	8	11	18	21	28	31	38	41	48	51	58	61	68
n = 2	11	16	25	30	39	44	53	58	67	72	81	86	95
n = 3	18	25	40	47	62	69	84	91	106	113	128	135	150
n = 4	21	30	47	56	73	82	99	108	125	134	151	160	177
n = 5	28	39	62	73	96	107	130	141	164	175	198	209	232
n = 6	31	44	69	82	107	120	145	158	183	196	221	234	259
n = 7	38	53	84	99	130	145	176	191	222	237	268	283	314
n = 8	41	58	91	108	141	158	191	208	241	258	291	308	341
n = 9	48	67	106	125	164	183	222	241	280	299	338	357	396
n = 10	51	72	113	134	175	196	237	258	299	320	361	382	423
n = 11	58	81	128	151	198	221	268	291	338	361	408	431	478
n = 12	61	86	139	164	217	242	295	320	373	398	451	476	529
n = 13	68	95	150	177	232	259	314	341	396	423	478	505	560

■ Composite Indices ■ Diagonal ■ Redundancy

**Table 3:** Extension of matrix in Table (2) for  $(1 \leq n < 25)$  and  $(14 \leq m \leq 25)$

m	14	15	16	17	18	19	20	21	22	23	24	25
n = 1	71	78	81	88	91	98	101	108	111	118	121	128
n = 2	100	109	114	123	128	137	142	151	156	165	170	179
n = 3	157	172	179	194	201	216	223	238	245	260	267	282
n = 4	186	203	212	229	238	255	264	281	290	307	316	333
n = 5	243	266	277	300	311	334	345	368	379	402	413	436
n = 6	272	297	310	335	348	373	386	411	424	449	462	487
n = 7	329	360	375	406	421	452	467	498	513	544	559	590
n = 8	358	391	408	441	458	491	508	541	558	591	608	641
n = 9	415	454	473	512	531	570	589	628	647	686	705	744
n = 10	444	485	506	547	568	609	630	671	692	733	754	795
n = 11	501	548	571	618	641	688	711	758	781	828	851	898
n = 12	554	607	632	685	710	763	782	831	856	905	930	979
n = 13	587	642	669	724	751	806	833	888	915	970	997	1052
n = 14	616	673	702	759	788	845	874	931	960	1017	1046	1103
n = 15	673	736	767	830	861	924	955	1018	1049	1112	1143	1206
n = 16	702	767	800	865	898	963	996	1061	1094	1159	1192	1257
n = 17	759	830	865	936	971	1042	1077	1148	1183	1254	1289	1360
n = 18	788	861	898	971	1008	1081	1118	1191	1228	1301	1338	1411
n = 19	845	924	963	1042	1081	1160	1199	1278	1317	1396	1435	1514
n = 20	874	955	996	1077	1118	1199	1240	1321	1362	1443	1484	1565
n = 21	931	1018	1061	1148	1191	1278	1321	1408	1451	1538	1581	1668
n = 22	960	1049	1094	1183	1228	1317	1362	1451	1496	1585	1630	1719
n = 23	1017	1112	1159	1254	1301	1396	1443	1538	1585	1680	1727	1822
n = 24	1046	1143	1192	1289	1338	1435	1484	1581	1630	1727	1776	1873
n = 25	1103	1206	1257	1360	1411	1514	1565	1668	1719	1822	1873	1976



If a single equation for Equation 39 is desired, ( $i_{(not\ prime)}$ ) can be written as shown in Equation 40:

$$\begin{aligned}
 (i_{(not\ prime)}) &= \left[ \begin{aligned} &(ma(n) + n)\cos^2\left(\frac{\pi m}{2}\right) \\ &+ (ma(m) + 2n + 1)\sin^2\left(\frac{\pi m}{2}\right) \end{aligned} \right] \sin^2\left(\frac{\pi n}{2}\right) \\
 &+ \left[ \begin{aligned} &(ma(n) + n)\cos^2\left(\frac{\pi m}{2}\right) \\ &+ (ma(n) + 2n)\sin^2\left(\frac{\pi m}{2}\right) \end{aligned} \right] \cos^2\left(\frac{\pi n}{2}\right)
 \end{aligned} \tag{40}$$

By using the trigonometric identity in Equation 13, Equation 40 yields Equation (41):

$$\begin{aligned}
 (i_{(not\ prime)}) &= ma(n) + n + n\sin^2\left(\frac{\pi m}{2}\right) \\
 &+ \sin^2\left(\frac{\pi m}{2}\right)\sin^2\left(\frac{\pi n}{2}\right)
 \end{aligned} \tag{41}$$

Using Equation 14 in place of  $a(n)$  in Equation 41 results in Equation 38 and the same equation for ( $i_{(not\ prime)}$ ) has been obtained by two different methods, which validates a mathematical connection between ( $i_{(not\ prime)}$ ) and the ( $a$ ) function. The new form of  $P(n,m)$  is shown in Equation 42:

$$P(n,m) = \left\{ \begin{aligned} &a_{prime}(n) = 3n + 1 + \sin^2\left(\frac{\pi n}{2}\right) \\ &i_{(not\ prime)} = 3nm + m + m\sin^2\left(\frac{\pi n}{2}\right) + n + n\sin^2\left(\frac{\pi n}{2}\right) \\ &\quad + \sin^2\left(\frac{\pi m}{2}\right)\sin^2\left(\frac{\pi n}{2}\right) \\ &1 \leq m \\ &1 \leq n \leq N \\ &1 \leq i_{(not\ prime)} \leq N \\ &n \neq i_{(not\ prime)} \end{aligned} \right. \tag{42}$$

Equation 42 possesses a form of ( $i_{(not\ prime)}$ ), which can be implemented without excessively generating redundant values. The current values of ( $i_{(not\ prime)}$ ) no longer depend on any previous values of ( $i_{(not\ prime)}$ ), which enables all redundant values to be avoided, making the entire process more efficient.

### Conclusion

The results presented in this study further validate the notion that there exists an equation (the  $a$ ) function that will generate all of the prime numbers, starting with 5, but it also generates composites periodically that are not divisible by 2 or 3. Thus, there also exists an equation for

eliminating the composites. Through algebraic and trigonometric manipulations, both equations can be significantly simplified making the entire process more efficient. The process becomes more efficient because redundancy in the ( $i_{(not\ prime)}$ ) equation becomes more avoidable, since more flexibility is established in choosing which values of the ( $i_{(not\ prime)}$ ) need to be calculated and which values are redundant. The procedures presented in this work, outline the general process for identifying and avoiding the remaining redundancy in ( $i_{(not\ prime)}$ ), but more experimentation with multiple, more elaborate computer systems is needed to generate higher order prime numbers and further validate the whole process.

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### Author's Contributions

The primary author, Jonathan M. Dugas, discovered that the ( $D$ ) component of the ( $i_{(not\ prime)}$ ) equation is quadratic in nature and the remaining components of the ( $i_{(not\ prime)}$ ) equation and the ( $a$ ) function are linear in nature. Upon making these discoveries, the primary author consulted the secondary, Brian M. O'Connor. The secondary author helped find methods for modifying the form of the equations in this work. Both authors then used those methods to simplify the components comprising the prime number generator. The primary author composed the paper and wrote the computer code needed to validate the simplified prime generator and the secondary author helped direct the mathematical procedures and language used to compose this work.

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