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A New Bivariate Distribution with Generalized Quadratic Hazard Rate Marginals

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Abstract: Recently, Sarhan (2009) introduced a new distribution named generalized quadratic hazard rate distribution. In this study, we define a bivariate Generalized Quadratic Hazard Rate Distribution (BGQHRD). The joint cumulative distribution function and joint survival function are derived in compact forms. Several properties of BGQFRD have been discussed. The conditional probability density function, *r*th moments and joint and marginal moment generating functions are obtained. Parameters estimators using the maximum likelihood method are obtained. A numerical illustration is used to obtain maximum likelihood estimators (MLEs). Moreover, we study the behavior of the estimators numerically.

Keywords: Bivariate Distributions, Generalized Quadratic Hazard Rate Distribution, Maximum Likelihood Estimators, Information Matrix

Introduction

Recently, Sarhan (2009) introduced a new distribution named generalized quadratic hazard rate distribution. This paper introduces a Bivariate Generalized Quadratic Hazard Rate Distribution (BGQHRD) by using the method of Marshall and Olkin (1986).

The Generalized Quadratic Hazard Rate Distribution (GQHRD), generalizes several distributions such as the quadratic hazard rate, the generalized linear failure rate, the generalized exponential and the generalized Rayleigh distributions.

The GQHRD may have an decreasing (increasing) hazard or a bathtub shaped hazard or an upside-down bathtub shaped hazard function. This property enables this distribution to be used in many applications such as in reliability, life testing, survival analysis.

Sarhan and Balakrishnan (2007) discussed Marshall and Olkin bivariate exponential distribution, Al-Khedhairi and El-Gohary (2008) introduced a new class of bivariate Gompertz distributions, Kundu and Gupta (2009) studied the bivariate generalized exponential distribution, El-Sherpieny *et al.* (2013) expressed a new bivariate generalized Gompertz distribution and Kundu and Gupta (2013) presented Marshall-Olkin bivariate Weibull distribution.

The random variable X has the Quadratic Hazard Rate Distribution (QHRD) with parameters *a*, *b*, *c* if its Cumulative Distribution Function (CDF) is:

$$F(x;a,b,c) = 1 - \exp\left\{-\left(ax + \frac{b}{2}x^2 + \frac{c}{3}x^3\right)\right\} \quad (1)$$

where, $a \geq 0$, $c \geq 0$ and $b \geq -2\sqrt{ac}$. This restriction on the parameter space is made to be insure that the hazard function for QHRD is positive, see Bain (1974):

$$h(x;a,b,c) = a + bx + cx^2, \; x \geq 0 \quad (2)$$

The exponential distribution (ED(a)) can be obtained from QHRD(a,b,c) when $b = 0$, $c = 0$. We can get the Rayleigh distribution (RD(b)) from QHRD when $a = 0$, $c = 0$, the Weibull with shape parameter equals 3 (WD(c,3)) can be concluded from QHRD(a,b,c) when $a = 0$, $b = 0$ and linear hazard rate distribution (LFRD(a,b)) can be derived from QHRD(a,b,c) when $c = 0$.

Sarhan (2009) introduced the generalized quadratic hazard rate distribution with parameters a, b, c and d, (GQHRD(a,b,c,d)). The GQHRD(a,b,c,d) has the following CDF:

$$F(x;a,b,c) = \left[1 - \exp\left\{-\left(ax + \frac{b}{2}x^2 + \frac{c}{3}x^3\right)\right\}\right]^d, \; x \geq 0 \quad (3)$$

where, $a \geq 0$, $c \geq 0$ and $b \geq -2\sqrt{ac}$.

The main objective of this article is to introduce and study the bivariate generalized quadratic hazard rate distribution. The BGQHRD generalizes the following distributions:

- The bivariate generalized LFRD when $c = 0$
- The bivariate generalized ED, when $b = 0$, $c = 0$, $a>0$
- The bivariate generalized RD, when $a = 0$, $c = 0$, $b>0$

This paper can be organized as follows. Some properties of the BGQHRD are introduced in section 2. Section 3 gives the mathematical expectations. Section 4 presents the moment generating functions. The parameter estimations using maximum likelihood is given in section 5. A set of real data is used as an application in section 6.

A New Bivariate Generalized Quadratic Hazard Rate Distribution

In this section, we discuss the BGQHRD. We start with the joint cumulative distribution function of the distribution.

Let X be a random variable has GQHR distribution with parameters a, b, c and d, if its PDF and CDF, respectively, is:

$$f(x;a,b,c,\alpha) = (a+bx+cx^2)e^{-\left(ax+\frac{b}{2}x^2+\frac{c}{3}x^3\right)} \times \left[1-e^{-\left(ax+\frac{b}{2}x^2+\frac{c}{3}x^3\right)}\right]^{\alpha-1} \quad (4)$$

$$F(x;a,b,c,\alpha) = \left[1-e^{-\left(ax+\frac{b}{2}x^2+\frac{c}{3}x^3\right)}\right]^{\alpha} \quad (5)$$

where, $\alpha>$, $a\geq 0$, $c\geq 0$ and $b \geq -2\sqrt{ac}$.

The Joint Cumulative Distribution Function

Suppose that U_1~GQHRD(α_1,a,b,c), U_2~GQHRD(α_2, a,b,c) and U_3~GQHRD(α_3,a,b,c) are independently distributed.

Define $X_1 = \max(U_1, U_3)$ and $X_2 = \max(U_2, U_3)$. The bivariate vector (X_1, X_2) has a BGQHRD with parameters α_1, α_2, α_3, a, b and c.

Lemma 1

The joint cumulative distribution function of (X_1, X_2) is given as:

$$F_{X_1,X_2}(x_1,x_2) = \left[1-e^{-\left(ax_1+\frac{b}{2}x_1^2+\frac{c}{3}x_1^3\right)}\right]^{\alpha_1}\left[1-e^{-\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)}\right]^{\alpha_2} \times \left[1-e^{-\left(az+\frac{b}{2}z^2+\frac{c}{3}z^3\right)}\right]^{\alpha_3} \quad (6)$$

where, $z = \min(x_1, x_2)$.

Proof

Since $F(x_1,x_2) = P[X_1\leq x_1, X_2\leq x_2]$, we get:

$$\begin{aligned} F(x_1,x_2) &= P\left[\max(U_1,U_3)\leq x_1, \max(U_2,U_3)\leq x_2\right] \\ &= P\left[U_1\leq x_1, U_2\leq x_2, U_3\leq \min(x_1,x_2)\right] \end{aligned}$$

As U_i, $i = 1,2,3$ are mutually independent, we obtain:

$$\begin{aligned} F(x_1,x_2) &= P\left[U_1\leq x_1\right]P\left[U_2\leq x_2\right]P\left[U_3\leq \min(x_1,x_2)\right]\perp \\ &= F_{GQHR}(x_1;\alpha_1,a,b,c)F_{GQHR}(x_2;\alpha_2,a,b,c)F_{GQHR}(z;\alpha_3,a,b,c). \end{aligned} \quad (7)$$

Substituting from Equation 5 into Equation 7, we obtain Equation 6. This completes the proof.

The Joint Probability Density Function

The following theorem gives the joint PDF of the BGQHRD.

Theorem 2

If (X_1, X_2) has BGQHRD, then the joint probability density function of (X_1, X_2) is given by:

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} f_1(x_1,x_2) & \text{if } x_1 < x_2, \\ f_2(x_1,x_2) & \text{if } x_2 < x_1, \\ f_3(x,x) & \text{if } x_1 = x_2 = x. \end{cases} \quad (8)$$

where:

$$f_1(x_1,x_2) = (\alpha_1+\alpha_3)\left(a+bx_1+cx_1^2\right)e^{-\left(ax_1+\frac{b}{2}x_1^2+\frac{c}{3}x_1^3\right)} \left[1-e^{-\left(ax_1+\frac{b}{2}x_1^2+\frac{c}{3}x_1^3\right)}\right]^{\alpha_1+\alpha_3-1}\alpha_2\left(a+bx_2+cx_2^2\right) e^{-\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)}\left[1-e^{-\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)}\right]^{\alpha_2-1} \quad (9)$$

$$f_2(x_1,x_2) = \alpha_1\left(a+bx_1+cx_1^2\right)e^{-\left(ax_1+\frac{b}{2}x_1^2+\frac{c}{3}x_1^3\right)}\left[1-e^{-\left(ax_1+\frac{b}{2}x_1^2+\frac{c}{3}x_1^3\right)}\right]^{\alpha_1-1}(\alpha_2+\alpha_3)\left(a+bx_2+cx_2^2\right)e^{-\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)}\left[1-e^{-\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)}\right]^{\alpha_2+\alpha_3-1} \quad (10)$$

$$f_3(x,x) = \alpha_3\left(a+bx+cx^2\right)e^{-\left(ax+\frac{b}{2}x^2+\frac{c}{3}x^3\right)} \left[1-e^{-\left(ax+\frac{b}{2}x^2+\frac{c}{3}x^3\right)}\right]^{\alpha_1+\alpha_2+\alpha_3-1}. \tag{11}$$

Proof

Let first assume that $x_1<x_2$, then $F(x_1,x_2)$ in Equation 6 will be denoted by $F_1(x_1,x_2)$ and becomes:

$$F_1(x_1,x_2) = \left[1-e^{-\left(ax_1+\frac{b}{2}x_1^2+\frac{c}{3}x_1^3\right)}\right]^{\alpha_1+\alpha_3}\left[1-e^{-\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)}\right]^{\alpha_2}$$

Differentiating $F_1(x_1,x_2)$ with respect to x_1 and x_2, we obtain the expression of $f_1(x_1,x_2)$ as given in Equation 9. By the same way we can obtain $f_2(x_1,x_2)$ when $x_2<x_1$. But $f_3(x_1,x_2)$ cannot be obtained in a similar way. We can use the following identity to derive $f_3(x,x)$ as:

$$\int_0^\infty\int_0^{x_2} f_1(x_1,x_2)dx_1dx_2 + \int_0^\infty\int_0^{x_1} f_2(x_1,x_2)dx_2dx_1 + \int_0^\infty f_3(x,x)dx = 1$$

Let:

$$I_1 = \int_0^\infty\int_0^{x_2} f_1(x_1,x_2)dx_1dx_2 \text{ and } I_2 = \int_0^\infty\int_0^{x_1} f_2(x_1,x_2)dx_2dx_1$$

Then:

$$\begin{aligned} I_1 &= \int_0^\infty\int_0^{x_2}(\alpha_1+\alpha_3)\left(a+bx_1+cx_1^2\right)e^{-\left(ax_1+\frac{b}{2}x_1^2+\frac{c}{3}x_1^3\right)} \\ &\times\left[1-e^{-\left(ax_1+\frac{b}{2}x_1^2+\frac{c}{3}x_1^3\right)}\right]^{\alpha_1+\alpha_3-1}\alpha_2\left(a+bx_2+cx_2^2\right) \\ &e^{-\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)}\left[1-e^{-\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)}\right]^{\alpha_2-1}dx_1dx_2 \\ &= \int_0^\infty \alpha_2\left(a+bx_2+cx_2^2\right)e^{-\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)} \\ &\left[1-e^{-\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)}\right]^{\alpha_1+\alpha_2+\alpha_3-1}dx_2 \end{aligned} \tag{12}$$

Similarly:

$$I_2 = \int_0^\infty \alpha_1\left(a+bx_1+cx_1^2\right)e^{-\left(ax_1+\frac{b}{2}x_1^2+\frac{c}{3}x_1^3\right)} \left[1-e^{-\left(ax_1+\frac{b}{2}x_1^2+\frac{c}{3}x_1^3\right)}\right]^{\alpha_1+\alpha_2+\alpha_3-1}dx_1. \tag{13}$$

From Equation 12 and 13, we then get:

$$\begin{aligned} \int_0^\infty f_3(x,x)dx &= \int_0^\infty\left(\alpha_1+\alpha_2+\alpha_3\right)(a+bx+cx^2)e^{-\left(ax+\frac{b}{2}x^2+\frac{c}{3}x^3\right)} \\ &\left[1-e^{-\left(ax+\frac{b}{2}x^2+\frac{c}{3}x^3\right)}\right]^{\alpha_1+\alpha_2+\alpha_3-1}dx - \int_0^\infty \alpha_2(a+bx+cx^2) \\ &e^{-\left(ax+\frac{b}{2}x^2+\frac{c}{3}x^3\right)}\left[1-e^{-\left(ax+\frac{b}{2}x^2+\frac{c}{3}x^3\right)}\right]^{\alpha_1+\alpha_2+\alpha_3-1}dx \\ &-\int_0^\infty \alpha_1\left(a+bx_1+cx_1^2\right)e^{-\left(ax+\frac{b}{2}x^2+\frac{c}{3}x^3\right)}\left[1-e^{-\left(ax+\frac{b}{2}x^2+\frac{c}{3}x^3\right)}\right]^{\alpha_1+\alpha_2+\alpha_3-1}dx \\ &= \int_0^\infty \alpha_3\left(a+bx_1+cx_1^2\right)e^{-\left(ax+\frac{b}{2}x^2+\frac{c}{3}x^3\right)}\left[1-e^{-\left(ax+\frac{b}{2}x^2+\frac{c}{3}x^3\right)}\right]^{\alpha_1+\alpha_2+\alpha_3-1}dx. \end{aligned}$$

That is:

$$\begin{aligned} &f_3(x,x) \\ &= \alpha_3\left(a+bx_1+cx_1^2\right)e^{-\left(ax+\frac{b}{2}x^2+\frac{c}{3}x^3\right)}\left[1-e^{-\left(ax+\frac{b}{2}x^2+\frac{c}{3}x^3\right)}\right]^{\alpha_1+\alpha_2+\alpha_3-1} \end{aligned}$$

Hence the proof of the theorem is completed.

Marginal Probability Density Function

The following theorem gives the marginal PDF of X_i, $i = 1, 2$.

Theorem 3

The marginal probability density function of X_i ($i = 1, 2$) can be derived as follows:

$$f_{X_i}(x_i) = (\alpha_i+\alpha_3)\left(a+bx_i+cx_i^2\right)e^{-\left(ax_i+\frac{b}{2}x_i^2+\frac{c}{3}x_i^3\right)} \left[1-e^{-\left(ax_i+\frac{b}{2}x_i^2+\frac{c}{3}x_i^3\right)}\right]^{\alpha_i+\alpha_3-1} \tag{14}$$

where, $i = 1,2$.

Proof

Let $F_{X_i}(x_i)$ be the CDF of X_i ($i = 1, 2$), given by:

$$\begin{aligned} F_{X_i}(x_i) &= P\left[X_i \le x_i\right] = P\left[\max(U_i,U_3) \le x_i\right] \\ &= P\left[U_i \le x_i, U_3 \le x_i\right] \end{aligned}$$

Since U_i and U_3 are independent, we have:

$$F_{X_i}(x_i) = P\left[U_i \le x_i\right]P\left[U_3 \le x_i\right] = \left[1-e^{-\left(ax_i+\frac{b}{2}x_i^2+\frac{c}{3}x_i^3\right)}\right]^{\alpha_i+\alpha_3}$$

Differentiating with respect to x_i, we derive the formula given in Equation 14.

That is, $X_i \sim GQHR(\alpha_i+\alpha_3,a,b,c)$.

Remark 4

The relation between joint survival function, marginals and the joint CDF is given by:

$$S_{X_1,X_2}(x_1,x_2) = 1 - F_{X_1}(x_1) - F_{X_2}(x_2) + F_{X_1,X_2}(x_1,x_2) \quad (15)$$

Therefore, the joint survival function of X_1 and X_2 also can be derived.

Lemma 5

If (X_1,X_2) have bivariate generalized quadratic hazard rate distribution, then $\max(X_1,X_2) \sim$ GQHR($\alpha_1+\alpha_2+\alpha_3$,a,b,c).

Proof:

$$\begin{aligned} F_{\max}(x) &= P[\max(X_1,X_2) \le x] = P[X_1 \le x, X_2 \le x] \\ &= P[U_1 \le x, U_2 \le x, U_3 \le x] \\ &= \left[1 - e^{-\left(ax + \frac{b}{2}x^2 + \frac{c}{3}x^3\right)}\right]^{\alpha_1+\alpha_2+\alpha_3} \end{aligned} \quad (16)$$

That is the maximum of X_1 and X_2 is also generalized quadratic hazard rate.

The Conditional Probability Density Function

Since the marginal probability density function of X_i (i = 1, 2) is derived, we can now find the conditional probability density function.

Theorem 6

The conditional probability density function of X_i, given $X_j = x_j$ ($i \ne j$ = 1, 2), is obtained by:

$$f_{X_i|X_j}(x_i \mid x_j) = \begin{cases} f^{(1)}_{X_i|X_j}(x_i \mid x_j) & \text{if } x_i < x_j \\ f^{(2)}_{X_i|X_j}(x_i \mid x_j) & \text{if } x_j < x_i \\ f^{(3)}_{X_i|X_j}(x_i \mid x_j) & \text{if } x_i = x_j. \end{cases} \quad (17)$$

where:

$$\begin{aligned} f^{(1)}_{X_i|X_j}(x_i \mid x_j) &= \alpha_j(\alpha_i+\alpha_3)\left(a + bx_i + cx_i^2\right) e^{-\left(ax_i + \frac{b}{2}x_i^2 + \frac{c}{3}x_i^3\right)} \\ &\left[1 - e^{-\left(ax_i + \frac{b}{2}x_i^2 + \frac{c}{3}x_i^3\right)}\right]^{\alpha_i+\alpha_3-1} \left\{(\alpha_j+\alpha_3)\left[1 - e^{-\left(ax_j + \frac{b}{2}x_j^2 + \frac{c}{3}x_j^3\right)}\right]^{\alpha_3}\right\}^{-1}, \\ f^{(2)}_{X_i|X_j}(x_i \mid x_j) &= \alpha_i\left(a + bx_i + cx_i^2\right) e^{-\left(ax_i + \frac{b}{2}x_i^2 + \frac{c}{3}x_i^3\right)} \\ &\left[1 - e^{-\left(ax_i + \frac{b}{2}x_i^2 + \frac{c}{3}x_i^3\right)}\right]^{\alpha_i+-1}, \\ f^{(3)}_{X_i|X_j}(x_i \mid x_j) &= \frac{\alpha_3}{\alpha_j+\alpha_3}\left[1 - e^{-\left(ax_i + \frac{b}{2}x_i^2 + \frac{c}{3}x_i^3\right)}\right]^{\alpha_i} \end{aligned} \quad (18)$$

Proof

Since:

$$f_{X_i|X_j}(x_i \mid x_j) = \frac{f_{X_i,X_j}(x_i,x_j)}{f_{X_j}(x_j)}$$

By substituting from Equation 8 and 14, in the above relation, the theorem follows directly.

Mathematical Expectations

We can derive the μ'_r (rth moments about zero) of X_i, $E[X_1X_2]$ and $E[X_i|X_j]$, ($i \ne j$ = 1, 2), based on the results introduced in the last two subsections.

Theorem 7

The rth moments of X_i (i = 1,2) about zero is given by:

$$\begin{aligned} \mu'_r = E[X_i^r] &= (\alpha_i+\alpha_3)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}(-1)^{j+k+\ell}\binom{\alpha_i+\alpha_3-1}{j} \\ &\frac{(j+1)^{k+\ell}b^k c^\ell}{2^k 3^\ell k!\ell!}\Gamma^1_{j,k,\ell}, \end{aligned} \quad (19)$$

where:

$$\begin{aligned} \Gamma^1_{j,k,\ell} &= \frac{a\Gamma(2k+3\ell+r+1)}{[(j+1)a]^{2k+3\ell+r+1}} + \frac{b\Gamma(2k+3\ell+r+2)}{[(j+1)a]^{2k+3\ell+r+2}} \\ &+ \frac{c\Gamma(2k+3\ell+r+3)}{[(j+1)a]^{2k+3\ell+r+3}}. \end{aligned}$$

Proof

Since the rth moments is defined by:

$$\mu'_r = E[X_i^r] = \int_0^\infty x_i^r f_{X_i}(x_i)\,dx_i,$$

Substituting for $f_{Xi}(x_i)$ from Equation 14, we get:

$$\begin{aligned} E(X_i^r) &= (\alpha_i+\alpha_3)\int_0^\infty x_i^r\left(a + bx_i + cx_i^2\right)e^{-\left(ax_i + \frac{b}{2}x_i^2 + \frac{c}{3}x_i^3\right)} \\ &\left[1 - e^{-\left(ax_i + \frac{b}{2}x_i^2 + \frac{c}{3}x_i^3\right)}\right]^{\alpha_i+\alpha_3-1} dx_i \end{aligned}$$

Since $0 < e^{-\left(ax_i + \frac{b}{2}x_i^2 + \frac{c}{3}x_i^3\right)} < 1$, then using the binomial expansion of $\left[1 - e^{-\left(ax_i + \frac{b}{2}x_i^2 + \frac{c}{3}x_i^3\right)}\right]^{\alpha_i+\alpha_3-1}$ given by:

$$\left[1 - e^{-\left(ax_i + \frac{b}{2}x_i^2 + \frac{c}{3}x_i^3\right)}\right]^{\alpha_i+\alpha_3-1} = \sum_{j=0}^{\infty}(-1)^j\binom{\alpha_i+\alpha_3-1}{j}e^{-j\left(ax_i + \frac{b}{2}x_i^2 + \frac{c}{3}x_i^3\right)}$$

Then:

$$E(X_i^r)=\left(\alpha_i+\alpha_3\right)\sum_{j=0}^{\infty}(-1)^j\binom{\alpha_i+\alpha_3-1}{j}\times\int_0^{\infty}x_i^r\left(a+bx_i+cx_i^2\right)e^{-(j+1)\left(ax_i+\frac{b}{2}x_i^2+\frac{c}{3}x_i^3\right)}dx_i. \quad (20)$$

But the expansion of $e^{-(j+1)\frac{b}{2}x_i^2}$ and $e^{-(j+1)\frac{c}{3}x_i^3}$ are given by:

$$e^{-(j+1)\frac{b}{2}x_i^2}=\sum_{k=0}^{\infty}\frac{(-1)^k(j+1)^k b^k x_i^{2k}}{2^k k!}\ and\quad e^{-(j+1)\frac{c}{3}x_i^3}=\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}(j+1)^{\ell}c^{\ell}x_i^{3\ell}}{3^{\ell}\ell!}, \quad (21)$$

Substituting from Equation 21 into Equation 20, we have:

$$E(X_i^r)=\left(\alpha_i+\alpha_3\right)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}(-1)^{j+k+\ell}\binom{\alpha_i+\alpha_3-1}{j}\times\frac{(j+1)^{k+\ell}b^k c^{\ell}}{2^k 3^{\ell}k!\ell!}\int_0^{\infty}\left(a+bx_i+cx_i^2\right)x_i^{2k+3\ell+r}e^{-(j+1)ax_i}dx_i$$
$$=\left(\alpha_i+\alpha_3\right)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}(-1)^{j+k+\ell}\binom{\alpha_i+\alpha_3-1}{j}\times\frac{(j+1)^{k+\ell}b^k c^{\ell}}{2^k 3^{\ell}k!\ell!}\Gamma^1_{j,k,\ell}, \quad (22)$$

where:

$$\Gamma^1_{j,k,\ell}=\int_0^{\infty}\left(a+bx_i+cx_i^2\right)x_i^{2k+3\ell+r}e^{-(j+1)ax_i}dx_i$$

Let $(j+1)ax_i=u\Rightarrow x_i=\frac{u}{(j+1)a}$, , then:

$$\Gamma^1_{j,k,\ell}=\frac{a\Gamma(2k+3\ell+r+1)}{\left[(j+1)a\right]^{2k+3\ell+r+1}}+\frac{b\Gamma(2k+3\ell+r+2)}{\left[(j+1)a\right]^{2k+3\ell+r+2}}+\frac{c\Gamma(2k+3\ell+r+3)}{\left[(j+1)a\right]^{2k+3\ell+r+3}} \quad (23)$$

where, $\Gamma(n)$, is the Gamma function.

Substituting from Equation 23 into Equation 22, we obtain the expression in Equation 19, which completes the proof.

Theorem 8

The expectation of X_1X_2 is given by:

$$E[X_1X_2]=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\alpha_2(\alpha_1+\alpha_3)\binom{\alpha_1+\alpha_3-1}{i}\binom{\alpha_2-1}{\ell}+\alpha_1(\alpha_2+\alpha_3)\binom{\alpha_2+\alpha_3-1}{i}\binom{\alpha_1-1}{\ell}\right]\beta_{i,j,k,\ell,m,n}+\alpha_3\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\alpha_1+\alpha_2+\alpha_3-1}{i}\Gamma^2_{i,j,k}, \quad (24)$$

where:

$$\Gamma^2_{i,j,k}=(-1)^{i+j+k}\frac{(i+1)^{j+k}b^j c^k}{2^j 3^k j!k!}\left[\frac{a\Gamma(2j+3k+3)}{\left[(i+1)a\right]^{2j+3k+3}}+\frac{b\Gamma(2j+3k+4)}{\left[(i+1)a\right]^{2j+3k+4}}+\frac{c\Gamma(2j+3k+5)}{\left[(i+1)a\right]^{2j+3k+5}}\right],$$

$$\beta_{i,j,k,r,\ell,m,n}=(-1)^{i+j+k+r+\ell+m+n}\frac{(i+1)^{j+k+r}(\ell+1)^{m+n}a^r b^{j+m}c^{k+n}}{2^{j+m}3^{k+n}j!k!r!m!n!}\left[\frac{a^2\Gamma(p+4)}{(p_1+2)\left[(\ell+1)a\right]^{p+4}}+\frac{ab(2p_1+5)\Gamma(p+5)}{(p_1+2)(p_1+3)\left[(\ell+1)a\right]^{p+5}}+\frac{ac(2p_1+6)\Gamma(p+6)}{(p_1+2)(p_1+4)\left[(\ell+1)a\right]^{p+6}}+\frac{b^2\Gamma(p+6)}{(p_1+3)\left[(\ell+1)a\right]^{p+6}}+\frac{bc(2p_1+7)\Gamma(p+7)}{(p_1+3)(p_1+4)\left[(\ell+1)a\right]^{p+7}}+\frac{c^2\Gamma(p+8)}{(p_1+4)\left[(\ell+1)a\right]^{p+8}}\right]$$

and $p = p_1+2m+3n$, $p_1= 2j+3k+r$.

Proof

Since:

$$E[X_1X_2]=\int_0^{\infty}\int_0^{\infty}x_1x_2f(x_1,x_2)dx_1dx_2=\int_0^{\infty}\int_0^{x_2}x_1x_2f_1(x_1,x_2)dx_1dx_2+\int_0^{\infty}\int_0^{x_1}x_1x_2f_2(x_1,x_2)dx_2dx_1+\int_0^{\infty}x^2f_3(x,x)dx_1dx_2. \quad (25)$$

Let:

$$I_1=\int_0^{\infty}\int_0^{x_2}x_1x_2f_1(x_1,x_2)dx_1dx_2,$$
$$I_2=\int_0^{\infty}\int_0^{x_1}x_1x_2f_2(x_1,x_2)dx_2dx_1$$
$$I_3=\int_0^{\infty}x^2f_3(x,x)dx_1dx_2.$$

From Equation 9, we have:

$$I_1=(\alpha_1+\alpha_3)\int_0^{\infty}\int_0^{x_2}x_1x_2\left(a+bx_1+cx_1^2\right)e^{-\left(ax_1+\frac{b}{2}x_1^2+\frac{c}{3}x_1^3\right)}\left[1-e^{-\left(ax_1+\frac{b}{2}x_1^2+\frac{c}{3}x_1^3\right)}\right]^{\alpha_1+\alpha_3-1}\alpha_2\left(a+bx_2+cx_2^2\right)e^{-\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)}\times\left[1-e^{-\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)}\right]^{\alpha_2-1}dx_1dx_2,$$

Since $0<e^{-\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)}<1$, by using the binomial expansion for $\left[1-e^{-\left(ax_1+\frac{b}{2}x_1^2+\frac{c}{3}x_1^3\right)}\right]^{\alpha_1+\alpha_3-1}$, we have:

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$$I_1 = (\alpha_1+\alpha_3)\sum_{i=0}^{\infty}\binom{\alpha_1+\alpha_3-1}{i}(-1)^i\int_0^\infty\int_0^{x_2} x_1x_2(a+bx_1+cx_1^2)e^{-(i+1)\left(ax_1+\frac{b}{2}x_1^2+\frac{c}{3}x_1^3\right)}$$
$$\alpha_2\left(a+bx_2+cx_2^2\right)e^{-\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)}\left[1-e^{-\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)}\right]^{\alpha_2-1}dx_1dx_2,$$

By using the expansions of $e^{-(i+1)ax_1}$, $e^{-(i+1)\frac{b}{2}x_1^2}$ and $e^{-(i+1)\frac{c}{3}x_1^3}$, we have:

$$I_1 = \alpha_2(\alpha_1+\alpha_3)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\binom{\alpha_1+\alpha_3-1}{i}(-1)^{i+j+k+r}$$
$$\frac{(i+1)^{j+k+r}b^jc^ka^r}{2^j3^kj!k!r!}\int_0^\infty\int_0^{x_2}x_1x_2(a+bx_1+cx_1^2)x_1^{2j+3k+r}$$
$$(a+bx_2+cx_2^2)e^{-\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)}\left[1-e^{-\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)}\right]^{\alpha_2-1}dx_1dx_2$$
$$=\alpha_2(\alpha_1+\alpha_3)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\binom{\alpha_1+\alpha_3-1}{i}(-1)^{i+j+k+r}\frac{(i+1)^{j+k+r}b^jc^ka^r}{2^j3^kj!k!r!}$$
$$\int_0^\infty x_2\left[\frac{ax_2^{2j+3k+r+2}}{2j+3k+r+2}+\frac{bx_2^{2j+3k+r+3}}{2j+3k+r+3}+\frac{cx_2^{2j+3k+r+4}}{2j+3k+r+4}\right] \quad (26)$$
$$(a+bx_2+cx_2^2)e^{-\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)}\left[1-e^{-\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)}\right]^{\alpha_2-1}dx_2$$
$$=\alpha_2(\alpha_1+\alpha_3)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\binom{\alpha_1+\alpha_3-1}{i}(-1)^{i+j+k+r}$$
$$\frac{(i+1)^{j+k+r}b^jc^ka^r}{2^j3^kj!k!r!}I_{11},$$

where:

$$I_{11}=\int_0^\infty x_2\left[\frac{ax_2^{2j+3k+r+2}}{2j+3k+r+2}+\frac{bx_2^{2j+3k+r+3}}{2j+3k+r+3}\right.$$
$$\left.+\frac{cx_2^{2j+3k+r+4}}{2j+3k+r+4}\right](a+bx_2+cx_2^2)\sum_{\ell=0}^{\infty}(-1)^\ell\binom{\alpha_2-1}{\ell}e^{-(\ell+1)\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)}dx_2$$
$$=\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{\ell+m+n}\binom{\alpha_2-1}{\ell}\frac{(\ell+1)^{m+n}b^mc^n}{2^m3^nm!n!}\int_0^\infty\left[\frac{ax_2^{2j+3k+r+3}}{2j+3k+r+2}\right.$$
$$\left.+\frac{bx_2^{2j+3k+r+4}}{2j+3k+r+3}+\frac{cx_2^{2j+3k+r+5}}{2j+3k+r+4}\right]$$
$$\left(ax_2^{2m+3n}+bx_2^{2m+3n+1}+cx_2^{2m+3n+2}\right)e^{-(\ell+1)ax_2}dx_2$$
$$=\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{\ell+m+n}\binom{\alpha_2-1}{l}\frac{(\ell+1)^{m+n}b^mc^n}{2^m3^nm!n!}\int_0^\infty\left[\frac{a^2x_2^{2j+3k+r+2m+3n+3}}{2j+3k+r+2}\right.$$
$$+\frac{abx_2^{2j+3k+r+2m+3n+4}}{2j+3k+r+3}+\frac{acx_2^{2j+3k+r+2m+3n+5}}{2j+3k+r+4}+\frac{abx_2^{2j+3k+r+2m+3n+4}}{2j+3k+r+2}$$
$$+\frac{b^2x_2^{2j+3k+r+2m+3n+5}}{2j+3k+r+3}+\frac{cbx_2^{2j+3k+r+2m+3n+6}}{2j+3k+r+4}+\frac{acx_2^{2j+3k+r+2m+3n+5}}{2j+3k+r+2}$$
$$\left.+\frac{bcx_2^{2j+3k+r+2m+3n+6}}{2j+3k+r+3}+\frac{c^2x_2^{2j+3k+r+2m+3n+7}}{2j+3k+r+4}\right]e^{-(\ell+1)ax_2}dx_2,$$

Setting $p_1= 2j+3k+r$, $p=p_1+2m+3n$ and $u=(l+1)ax_2$, then:

$$I_{11}=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{l+m+n}\binom{\alpha_2-1}{l}\frac{(l+1)^{m+n}b^mc^n}{2^m3^nm!n!}$$
$$\int_0^\infty\left[\frac{a^2u^{p+3}}{(p_1+2)[(l+1)a]^{p+4}}+\frac{ab(2p_1+5)u^{p+4}}{(p_1+2)(p_1+3)[(l+1)a]^{p+5}}\right.$$
$$+\frac{ac(2p_1+6)u^{p+5}}{(p_1+2)(p_1+4)[(\ell+1)a]^{p+6}}+\frac{b^2u^{p+5}}{(p_1+3)[(\ell+1)a]^{p+6}}$$
$$\left.+\frac{cb(2p_1+7)u^{p+6}}{(p_1+3)(p_1+4)[(\ell+1)a]^{p+7}}+\frac{c^2u^{p+7}}{(p_1+4)[(\ell+1)a]^{p+8}}\right]e^{-u}du \quad (27)$$
$$=\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{l+m+n}\binom{\alpha_2-1}{\ell}\frac{(\ell+1)^{m+n}b^mc^n}{2^m3^nm!n!}\left[\frac{a^2\Gamma(p+4)}{(p_1+2)[(\ell+1)a]^{p+4}}\right.$$
$$\frac{ab(2p_1+5)\Gamma(p+5)}{(p_1+2)(p_1+3)[(\ell+1)a]^{p+5}}+\frac{ac(2p_1+6)\Gamma(p+6)}{(p_1+2)(p_1+4)[(\ell+1)a]^{p+6}}$$
$$+\frac{b^2\Gamma(p+6)}{(p_1+3)[(\ell+1)a]^{p+6}}+\frac{cb(2p_1+7)\Gamma(p+7)}{(p_1+3)(p_1+4)[(\ell+1)a]^{p+7}}$$
$$\left.+\frac{c^2\Gamma(p+8)}{(p_1+4)[(\ell+1)a]^{p+8}}\right].$$

Substituting from Equation 27 into Equation 26 we have:

$$I_1=(\alpha_1+\alpha_3)\alpha_2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{\alpha_1+\alpha_3-1}{i}\binom{\alpha_2-1}{\ell}\beta_{i,j,k,r,\ell,m,n}, \quad (28)$$

where:

$$\beta_{i,j,k,r,l,m,n}=(-1)^{i+j+k+r+\ell+m+n}\frac{(i+1)^{j+k+r}(\ell+1)^{m+n}a^rb^{j+m}c^{k+n}}{2^{j+m}3^{k+n}j!k!r!m!n!}$$
$$\left[\frac{a^2\Gamma(p+4)}{(p_1+2)[(\ell+1)a]^{p+4}}+\frac{ab(2p_1+5)\Gamma(p+5)}{(p_1+2)(p_1+3)[(\ell+1)a]^{p+5}}\right.$$
$$+\frac{ac(2p_1+6)\Gamma(p+6)}{(p_1+2)(p_1+4)[(\ell+1)a]^{p+6}}+\frac{b^2\Gamma(p+6)}{(p_1+3)[(\ell+1)a]^{p+6}}$$
$$\left.\frac{cb(2p_1+7)\Gamma(p+7)}{(p_1+3)(p_1+4)[(\ell+1)a]^{p+7}}+\frac{c^2\Gamma(p+8)}{(p_1+4)[(\ell+1)a]^{p+8}}\right].$$

From Equation 10, we can find I_2, by the same way:

$$I_2=(\alpha_2+\alpha_3)\alpha_1\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{\alpha_2+\alpha_3-1}{i}\binom{\alpha_1-1}{\ell}\beta_{i,j,k,r,\ell,m,n} \quad (29)$$

From Equation 11, we have:

$$I_3 = \alpha_3 \int_0^\infty x^2 (a + bx + cx^2) e^{-\left(ax + \frac{b}{2}x^2 + \frac{c}{3}x^3\right)} \left[1 - e^{-\left(ax + \frac{b}{2}x^2 + \frac{c}{3}x^3\right)}\right]^{\alpha_1+\alpha_2+\alpha_3-1} dx,$$

Since $0 < e^{-\left(ax + \frac{b}{2}x^2 + \frac{c}{3}x^3\right)} < 1$, by using the binomial expansion for $\left[1 - e^{-\left(ax + \frac{b}{2}x^2 + \frac{c}{3}x^3\right)}\right]^{\alpha_1+\alpha_2+\alpha_3-1}$, we have:

$$I_3 = \alpha_3 \sum_{i=0}^\infty \binom{\alpha_1+\alpha_2+\alpha_3-1}{i} (-1)^i \int_0^\infty x^2 (a + bx + cx^2) e^{-(i+1)\left(ax + \frac{b}{2}x^2 + \frac{c}{3}x^3\right)} dx,$$

By using the Taylor's expression for $e^{-(i+1)\frac{b}{2}x^2}$ and $e^{-(i+1)\frac{c}{3}x^3}$, we have:

$$I_3 = \alpha_3 \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \binom{\alpha_1+\alpha_2+\alpha_3-1}{i} (-1)^{i+j+k} \frac{(i+1)^{j+k} b^j c^k}{2^j 3^k j! k!} \times \mathbb{S} \int_0^\infty (ax^2 + bx^3 + cx^4) x^{2j+3k} e^{-(i+1)ax} dx,$$

Put $u = (I + 1)ax$, we have:

$$I_3 = \alpha_3 \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \binom{\alpha_1+\alpha_2+\alpha_3-1}{i} (-1)^{i+j+k} \frac{(i+1)^{j+k} b^j c^k}{2^j 3^k j! k!} \times \int_0^\infty \left[\frac{au^{2j+3k+2}}{[(i+1)a]^{2j+3k+3}} + \frac{bu^{2j+3k+3}}{[(i+1)a]^{2j+3k+4}} + \frac{cu^{2j+3k+4}}{[(i+1)a]^{2j+3k+5}}\right] e^{-u} du \quad (30)$$
$$= \alpha_3 \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \binom{\alpha_1+\alpha_2+\alpha_3-1}{i} \Gamma_{i,j,k}^2,$$

where:

$$\Gamma_{i,j,k}^2 = (-1)^{i+j+k} \frac{(i+1)^{j+k} b^j c^k}{2^j 3^k j! k!} \left[\frac{a\Gamma(2j+3k+3)}{[(i+1)a]^{2j+3k+3}} + \frac{b\Gamma(2j+3k+4)}{[(i+1)a]^{2j+3k+4}} + \frac{c\Gamma(2j+3k+5)}{[(i+1)a]^{2j+3k+5}}\right].$$

Substituting from Equation 28-30 into Equation 25, we have Equation 24, this completes the proof.

From Theorem 7, setting $r = 1$, $i = 1, 2$ and Theorem 8, we can find the covariance between X_1, X_2:

$$Cov(X_1, X_2) = E[X_1 X_2] - E[X_1]E[X_2],$$

and:

$$Var(X_i) = E[X_i^2] - (E[X_i])^2.$$

Also, we can compute the correlation coefficient between X_1, X_2 as follows:

$$\rho_{X_1, X_2} = \frac{Cov(X_1, X_2)}{\sqrt{Var(X_1) Var(X_2)}}.$$

Theorem 9

The conditional expectation of X_i, given $X_j = x_j$ ($i \neq j = 1, 2$) is given by:

$$E[X_i \mid X_j = x_j] = \sum_{k=0}^\infty \sum_{\ell=0}^\infty \sum_{m=0}^\infty (-1)^{k+\ell+m} \frac{b^\ell c^m}{2^\ell 3^m \ell! m!} \Big\{ W_{i,j}(x_j) \times \binom{\alpha_i+\alpha_3-1}{k} (k+1)^{\ell+m} I_{x_j}^{(1)}(k,\ell,m) + \alpha_i \binom{\alpha_i-1}{k} (k+1)^{m+\ell} \quad (31)$$
$$\times I_{x_j}^{(2)}(k,\ell,m) + \frac{\alpha_3}{\alpha_j + \alpha_3} \binom{\alpha_i}{k} k^{\ell+m} \frac{\Gamma(2\ell+3m+2)}{(ka)^{2\ell+3m+2}} \Big\},$$

where:

$$W_{i,j}(x_j) = \frac{\alpha_j(\alpha_i + \alpha_3)}{\alpha_j + \alpha_3} \left[1 - e^{-(ax_j + \frac{b}{2}x_j^2 + \frac{c}{3}x_j^3)}\right]^{\alpha_3},$$
$$I_{x_j}^{(1)}(k,\ell,m) = \int_0^{(k+1)ax_j} J_{k,\ell,m}(u) du,$$
$$I_{x_j}^{(2)}(k,\ell,m) = \int_{(k+1)ax_j}^\infty J_{k,\ell,m}(u) du,$$
$$J_{k,\ell,m}(u) = \left[\frac{au^{2\ell+3m+1}}{[(k+1)a]^{2\ell+3m+2}} + \frac{bu^{2\ell+3m+2}}{[(k+1)a]^{2\ell+3m+3}} + \frac{cu^{2\ell+3m+3}}{[(k+1)a]^{2\ell+3m+4}}\right] e^{-u}.$$

Proof

Starting with:

$$E[X_i \mid X_j = x_j] = \int_0^\infty x_i f_{X_i|X_j}(x_i \mid x_j) dx_i$$
$$= \int_0^{x_j} x_i f_{X_i|X_j}^{(1)}(x_i \mid x_j) dx_i + \int_{x_j}^\infty x_i f_{X_i|X_j}^{(2)}(x_i \mid x_j) dx_i \quad (32)$$
$$+ \int_0^\infty x_i f_{X_i|X_j}^{(3)}(x_i \mid x_j) dx_i.$$

Substituting from Equation 17 into Equation 32, we obtain Equation 31.

Moment Generating Functions

In this section, we introduce the joint moment generating function of (X_1, X_2). The marginal moment generating function of X_i ($i = 1, 2$) also derived.

Lemma 10

The moment generating function of X_i ($i = 1, 2$) is given as follows:

$$M_{X_i}(t) = (\alpha_i+\alpha_3)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{\alpha_i+\alpha_3-1}{j}\frac{(-1)^{j+k+l}(j+1)^{k+l}b^k c^l}{2^k 3^l k!l!}$$
$$\times\left[\frac{a\Gamma(2k+3\ell+1)}{[(j+1)a+t]^{2k+3\ell+1}}+\frac{b\Gamma(2k+3\ell+2)}{[(j+1)a+t]^{2k+3\ell+2}}\right. \quad (33)$$
$$\left.+\frac{c\Gamma(2k+3\ell+3)}{[(j+1)a+t]^{2k+3\ell+3}}\right].$$

Proof

Since:

$$M_{X_i}(t) = E\left[e^{-tX_i}\right] = \int_0^\infty e^{-tx_i} f_{X_i}(x_i)dx_i,$$

and substituting for $f_{Xi}(x_i)$ from Equation 14, we get:

$$M_{X_i}(t) = (\alpha_i+\alpha_3)\int_0^\infty e^{-tx_i}\left(a+bx_i+cx_i^2\right)e^{-\left(ax_i+\frac{b}{2}x_i^2+\frac{c}{3}x_i^3\right)}$$
$$\left[1-e^{-\left(ax_i+\frac{b}{2}x_i^2+\frac{c}{3}x_i^3\right)}\right]^{\alpha_i+\alpha_3-1} dx_i,$$

From which we can derive the expression given in Equation 33.

Remark 11

The moment generating function $M_{Xi}(t)$ can be used to derive the marginal expectation of X_i as follows:

$$\mu_r' = E[X^r] = (-1)^r\left[\frac{d^r}{dt^r}M_{X_i}(t)\right]_{t=0}. \quad (34)$$

From Equation 33, we obtain:

$$\frac{d^r}{dt_i^r}M_{X_i}(t) = (-1)^r(\alpha_i+\alpha_3)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{\alpha_i+\alpha_3-1}{j}(-1)^{j+k+l}$$
$$\times\frac{(j+1)^{k+l}b^k c^l}{2^k 3^l k!l!}\left[\frac{a\Gamma(2k+3\ell+r+1)}{[(j+1)a+t]^{2k+3\ell+r+1}}+\frac{b\Gamma(2k+3\ell+r+2)}{[(j+1)a+t]^{2k+3\ell+r+2}}\right.$$
$$\left.+\frac{c\Gamma(2k+3\ell+r+3)}{[(j+1)a+t]^{2k+3\ell+r+3}}\right]$$

Therefore:

$$\left[\frac{d^r}{dt_i^r}M_{X_i}(t)\right]_{t=0} = (-1)^r(\alpha_i+\alpha_3)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{\alpha_i+\alpha_3-1}{j}(-1)^{j+k+l}$$
$$\times\frac{(j+1)^{k+l}b^k c^l}{2^k 3^l k!l!}\left[\frac{a\Gamma(2k+3l+r+1)}{[(j+1)a]^{2k+3l+r+1}}+\frac{b\Gamma(2k+3l+r+2)}{[(j+1)a]^{2k+3l+r+2}}\right. \quad (35)$$
$$\left.+\frac{c\Gamma(2k+3l+r+3)}{[(j+1)a]^{2k+3l+r+3}}\right].$$

Substituting from Equation 35 into Equation 34, we obtain μ_r' as given in Equation 19.

Theorem 12

The joint moment generating function of (X_1,X_2) is given as follows:

$$M_{X_1,X_2}(t_1,t_2)$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{\alpha_2(\alpha_1+\alpha_3)\binom{\alpha_1+\alpha_3-1}{i}\binom{\alpha_2-1}{\ell}\right.$$
$$\times\left[(i+1)a+t_1\right]^r F(\Gamma,t_2) \quad (36)$$
$$+\alpha_1(\alpha_2+\alpha_3)\binom{\alpha_2+\alpha_3-1}{i}\binom{\alpha_1-1}{\ell}\left[(i+1)a+t_2\right]^r$$
$$\left.F(\Gamma,t_1)\right\}+\alpha_3\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\alpha_1+\alpha_2+\alpha_3-1}{i}F(\Gamma,t_1,t_2),$$

where:

$$F(\Gamma,t_i) = (-1)^{i+j+k+r+\ell+m+n}\frac{(i+1)^{j+k}(\ell+1)^{m+n}b^{j+m}c^{k+n}}{2^{j+m}3^{k+n}j!k!r!m!n!}$$
$$\left[\frac{a^2\Gamma(p+2)}{(p_1+1)[(\ell+1)a+t_i]^{p+2}}+\frac{ab(2p_1+3)\Gamma(p+3)}{(p_1+1)(p_1+2)[(\ell+1)a+t_i]^{p+3}}\right.$$
$$+\frac{ac(2p_1+4)\Gamma(p+4)}{(p_1+1)(p_1+3)[(\ell+1)a+t_i]^{p+4}}+\frac{b^2\Gamma(p+4)}{(p_1+2)[(\ell+1)a+t_i]^{p+4}}$$
$$\left.+\frac{bc(2p_1+5)\Gamma(p+5)}{(p_1+2)(p_1+3)[(\ell+1)a+t_i]^{p+5}}+\frac{c^2\Gamma(p+6)}{(p_1+3)[(\ell+1)a+t_i]^{p+6}}\right],$$
$$i=1,2.$$
$$F(\Gamma,t_1,t_2) = (-1)^{i+j+k}\frac{(i+1)^{j+k}b^j c^k}{2^j 3^k j!k!}\left[\frac{a\Gamma(2j+3m+1)}{[(i+1)a+t_1+t_2]^{2j+3m+1}}\right.$$
$$\left.+\frac{b\Gamma(2j+3m+2)}{[(i+1)a+t_1+t_2]^{2j+3m+2}}+\frac{c\Gamma(2j+3m+3)}{[(i+1)a+t_1+t_2]^{2j+3m+3}}\right],$$

$p = 2m + 3n + p_1$, $p_1 = 2j + 3m + r$.

Proof

Since:

$$M(t_1,t_2) = E\left[e^{-(t_1X_1+t_2X_2)}\right] = \int_0^\infty\int_0^\infty e^{-(t_1x_1+t_2x_2)}f(x_1,x_2)dx_1dx_2$$
$$=\int_0^\infty\int_0^{x_2} e^{-(t_1x_1+t_2x_2)}f_1(x_1,x_2)dx_1dx_2 \quad (37)$$
$$+\int_0^\infty\int_0^{x_1} e^{-(t_1x_1+t_2x_2)}f_2(x_1,x_2)dx_2dx_1+\int_0^\infty e^{-(t_1+t_2)x}f_3(x,x)dx.$$

Let:

$$I_1 = \int_0^\infty\int_0^{x_2} e^{-(t_1x_1+t_2x_2)}f_1(x_1,x_2)dx_1dx_2,$$
$$I_2 = \int_0^\infty\int_0^{x_1} e^{-(t_1x_1+t_2x_2)}f_2(x_1,x_2)dx_2dx_1,$$
$$I_3 = \int_0^\infty e^{-(t_1+t_2)x}f_3(x,x)dx.$$

From Equation 9, we have:

$$I_1=(\alpha_1+\alpha_3)\int_0^{\infty}\int_0^{x_2}e^{-(t_1x_1+t_2x_2)}\left(a+bx_1+cx_1^2\right)e^{-\left(ax_1+\frac{b}{2}x_1^2+\frac{c}{3}x_1^3\right)}$$
$$\left[1-e^{-\left(ax_1+\frac{b}{2}x_1^2+\frac{c}{3}x_1^3\right)}\right]^{\alpha_1+\alpha_3-1}\alpha_2\left(a+bx_2+cx_2^2\right)e^{-\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)}$$
$$\left[1-e^{-\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)}\right]^{\alpha_2-1}dx_1dx_2.$$

Since $0<e^{-\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)}<1$, using the binomial expansion for $\left[1-e^{-\left(ax_1+\frac{b}{2}x_1^2+\frac{c}{3}x_1^3\right)}\right]^{\alpha_1+\alpha_3-1}$, we get:

$$I_1=(\alpha_1+\alpha_3)\sum_{i=0}^{\infty}\binom{\alpha_1+\alpha_3-1}{i}(-1)^i\int_0^{\infty}\int_0^{x_2}e^{-(t_1x_1+t_2x_2)}\left(a+bx_1+cx_1^2\right)$$
$$\times e^{-(i+1)\left(ax_1+\frac{b}{2}x_1^2+\frac{c}{3}x_1^3\right)}\alpha_2(a+bx_2+cx_2^2)e^{-\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)}$$
$$\left[1-e^{-\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)}\right]^{\alpha_2-1}dx_1dx_2$$

Using the Taylor's expression for $e^{-(i+1)\frac{b}{2}x_1^2}$, $e^{-(i+1)\frac{c}{3}x_1^3}$ and $e^{-[(i+1)a+t_1]x_1}$, we have:

$$I_1$$
$$=\alpha_2(\alpha_1+\alpha_3)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\binom{\alpha_1+\alpha_3-1}{i}(-1)^{i+j+k+r}\frac{(i+1)^{j+k}}{2^j3^k\,j!k!r!}$$
$$\times[(i+1)a+t_1]^r b^jc^k\int_0^{\infty}\int_0^{x_2}e^{-t_2x_2}(a+bx_1+cx_1^2)x_1^{2j+3k+r}$$
$$(a+bx_2+cx_2^2)e^{-\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)}\left[1-e^{-\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)}\right]^{\alpha_2-1}dx_1dx_2$$
$$=\alpha_2(\alpha_1+\alpha_3)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\binom{\alpha_1+\alpha_3-1}{i}(-1)^{i+j+k+r}\frac{(i+1)^{j+k}}{2^j3^k\,j!k!r!}$$
$$\times[(i+1)a+t_1]^r b^jc^k\int_0^{\infty}e^{-t_2x_2}\left[\frac{ax_2^{2j+3k+r+1}}{2j+3k+r+1}+\frac{bx_2^{2j+3k+r+2}}{2j+3k+r+2}\right.$$
$$\left.+\frac{cx_2^{2j+3k+r+3}}{2j+3k+r+3}\right]\left(a+bx_2+cx_2^2\right)e^{-\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)}$$
$$\left[1-e^{-\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)}\right]^{\alpha_2-1}dx_2$$
$$=\alpha_2(\alpha_1+\alpha_3)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\binom{\alpha_1+\alpha_3-1}{i}(-1)^{i+j+k+r}\frac{(i+1)^{j+k}}{2^j3^k\,j!k!r!}$$
$$\times[(i+1)a+t_1]^r b^jc^k I_{11}, \tag{38}$$

where:

$$I_{11}=\int_0^{\infty}e^{-t_2x_2}\left[\frac{ax_2^{2j+3k+r+1}}{2j+3k+r+1}+\frac{bx_2^{2j+3k+r+2}}{2j+3k+r+2}\right.$$
$$\left.+\frac{cx_2^{2j+3k+r+3}}{2j+3k+r+3}\right](a+bx_2+cx_2^2)e^{-\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)}$$
$$\times\left[1-e^{-\left(ax_2+\frac{b}{2}x_2^2+\frac{c}{3}x_2^3\right)}\right]^{\alpha_2-1}dx_2$$
$$=\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{\ell+m+n}\binom{\alpha_2-1}{\ell}\frac{(\ell+1)^{m+n}b^mc^n}{2^m3^n\,m!n!}\int_0^{\infty}\left[\frac{ax_2^{2j+3k+r+1}}{2j+3k+r+1}\right.$$
$$\left.+\frac{bx_2^{2j+3k+r+2}}{2j+3k+r+2}+\frac{cx_2^{2j+3k+r+3}}{2j+3k+r+3}\right]$$
$$\times\left(ax_2^{2m+3n}+bx_2^{2m+3n+1}+cx_2^{2m+3n+2}\right)e^{-[(l+1)a+t_2]x_2}dx_2$$
$$=\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{l+m+n}\binom{\alpha_2-1}{l}\frac{(\ell+1)^{m+n}b^mc^n}{2^m3^n\,m!n!}$$
$$\times\int_0^{\infty}\left[\frac{a^2x_2^{2j+3k+r+2m+3n+1}}{2j+3k+r+1}+\frac{abx_2^{2j+3k+r+2m+3n+2}}{2j+3k+r+2}\right.$$
$$+\frac{acx_2^{2j+3k+r+2m+3n+3}}{2j+3k+r+3}+\frac{abx_2^{2j+3k+r+2m+3n+2}}{2j+3k+r+1}$$
$$+\frac{b^2x_2^{2j+3k+r+2m+3n+3}}{2j+3k+r+2}+\frac{cbx_2^{2j+3k+r+2m+3n+4}}{2j+3k+r+3}$$
$$+\frac{acx_2^{2j+3k+r+2m+3n+3}}{2j+3k+r+1}+\frac{bcx_2^{2j+3k+r+2m+3n+4}}{2j+3k+r+2}$$
$$\left.+\frac{c^2x_2^{2j+3k+r+2m+3n+5}}{2j+3k+r+3}\right]e^{-[(\ell+1)a+t_2]x_2}dx_2,$$

Setting $p_1 = 2j + 3k + r$ and $p = p_1 + 2m + 3n$, $u=\left[(\ell+1)a+t_2\right]x_2$, we have:

$$I_{11}=\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{\ell+m+n}\binom{\alpha_2-1}{\ell}\frac{(\ell+1)^{m+n}b^mc^n}{2^m3^n\,m!n!}$$
$$\times\left[\frac{a^2\Gamma(p+2)}{(p_1+1)[(\ell+1)a+t_2]^{p+2}}+\frac{ab(2p_1+3)\Gamma(p+3)}{(p_1+1)(p_1+2)[(\ell+1)a+t_2]^{p+3}}\right.$$
$$+\frac{ac(2p_1+4)\Gamma(p+4)}{(p_1+1)(p_1+3)[(\ell+1)a+t_2]^{p+4}}+\frac{b^2\Gamma(p+4)}{(p_1+2)[(\ell+1)a+t_2]^{p+4}} \tag{39}$$
$$+\frac{cb(2p_1+5)\Gamma(p+5)}{(p_1+2)(p_1+3)[(\ell+1)a+t_2]^{p+5}}$$
$$\left.+\frac{c^2\Gamma(p+6)}{(p_1+3)[(\ell+1)a+t_2]^{p+6}}\right].$$

Substituting from Equation 39 into Equation 38 we have:

$$I_1=\alpha_2(\alpha_1+\alpha_3)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{\alpha_1+\alpha_3-1}{i}\binom{\alpha_2-1}{l}$$
$$\times[(i+1)a+t_1]^rF(\Gamma,t_2) \tag{40}$$

where:

$$F(\Gamma,t_2)=(-1)^{i+j+k+r+\ell+m+n}\frac{(i+1)^{j+k}(\ell+1)^{m+n}b^{j+m}c^{k+n}}{2^{j+m}3^{k+n}j!k!r!m!n!}$$
$$\times\Bigg[\frac{a^2\Gamma(p+2)}{(p_1+1)[(\ell+1)a+t_2]^{p+2}}+\frac{ab(2p_1+3)\Gamma(p+3)}{(p_1+1)(p_1+2)[(\ell+1)a+t_2]^{p+3}}$$
$$+\frac{ac(2p_1+4)\Gamma(p+4)}{(p_1+1)(p_1+3)[(\ell+1)a+t_2]^{p+4}}+\frac{b^2\Gamma(p+4)}{(p_1+2)[(\ell+1)a+t_2]^{p+4}}$$
$$+\frac{bc(2p_1+5)\Gamma(p+5)}{(p_1+2)(p_1+3)[(\ell+1)a+t_2]^{p+5}}+\frac{c^2\Gamma(p+6)}{(p_1+3)[(\ell+1)a+t_2]^{p+6}}\Bigg],$$

From Equation 10, we can find I_2, by the same way:

$$I_2=\alpha_1(\alpha_2+\alpha_3)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{\alpha_2+\alpha_3-1}{i}\binom{\alpha_1-1}{l}\times[(i+1)a+t_2]^rF(\Gamma,t_1) \quad (41)$$

where:

$$F(\Gamma,t_1)=(-1)^{i+j+k+r+\ell+m+n}\frac{(i+1)^{j+k}(\ell+1)^{m+n}b^{j+m}c^{k+n}}{2^{j+m}3^{k+n}j!k!r!m!n!}$$
$$\times\Bigg[\frac{a^2\Gamma(p+2)}{(p_1+1)[(\ell+1)a+t_1]^{p+2}}+\frac{ab(2p_1+3)\Gamma(p+3)}{(p_1+1)(p_1+2)[(\ell+1)a+t_1]^{p+3}}$$
$$+\frac{ac(2p_1+4)\Gamma(p+4)}{(p_1+1)(p_1+3)[(\ell+1)a+t_1]^{p+4}}+\frac{b^2\Gamma(p+4)}{(p_1+2)[(\ell+1)a+t_1]^{p+4}}$$
$$+\frac{bc(2p_1+5)\Gamma(p+5)}{(p_1+2)(p_1+3)[(\ell+1)a+t_1]^{p+5}}+\frac{c^2\Gamma(p+6)}{(p_1+3)[(\ell+1)a+t_1]^{p+6}}\Bigg].$$

From Equation 11, we have:

$$I_3=\alpha_3\int_0^\infty e^{-(t_1+t_2)x}(a+bx+cx^2)e^{-\left(ax+\frac{b}{2}x^2+\frac{c}{3}x^3\right)}\times\left[1-e^{-\left(ax+\frac{b}{2}x^2+\frac{c}{3}x^3\right)}\right]^{\alpha_1+\alpha_2+\alpha_3-1}dx.$$

Since $0<e^{-\left(ax+\frac{b}{2}x^2+\frac{c}{3}x^3\right)}<1$, by using the binomial expansion for $\left[1-e^{-\left(ax+\frac{b}{2}x^2+\frac{c}{3}x^3\right)}\right]^{\alpha_1+\alpha_2+\alpha_3-1}$, we have:

$$I_3=\alpha_3\sum_{i=0}^{\infty}\binom{\alpha_1+\alpha_2+\alpha_3-1}{i}(-1)^i\int_0^\infty e^{-(t_1+t_2)x}(a+bx+cx^2)\times e^{-(i+1)\left(ax+\frac{b}{2}x^2+\frac{c}{3}x^3\right)}dx,$$

By using the Taylor's expression for $e^{-(i+1)\frac{b}{2}x^2}$ and $e^{-(i+1)\frac{c}{3}x^3}$, we have:

$$I_3=\alpha_3\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\alpha_1+\alpha_2+\alpha_3-1}{i}(-1)^{i+j+k}\frac{(i+1)^{j+k}b^jc^k}{2^j3^kj!k!}\times\int_0^\infty(a+bx+cx^2)x^{2j+3k}e^{-[(i+1)a+t_1+t_2]x}dx,$$

Put $u=[(i+1)a+t_1+t_2]x$, then we have:

$$I_3=\alpha_3\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\alpha_1+\alpha_2+\alpha_3-1}{i}(-1)^{i+j+k}\frac{(i+1)^{j+k}b^jc^k}{2^j3^kj!k!}$$
$$\times\int_0^\infty\Bigg[\frac{au^{2j+3k}}{[(i+1)a+t_1+t_2]^{2j+3k+1}}+\frac{bu^{2j+3k+1}}{[(i+1)a+t_1+t_2]^{2j+3k+2}}+\frac{cu^{2j+3k+2}}{[(i+1)a+t_1+t_2]^{2j+3k+3}}\Bigg]e^{-u}du \quad (42)$$
$$=\alpha_3\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\alpha_1+\alpha_2+\alpha_3-1}{i}F(\Gamma,t_1,t_2),$$

where:

$$F(\Gamma,t_1,t_2)=(-1)^{i+j+k}\frac{(i+1)^{j+k}b^jc^k}{2^j3^kj!k!}\Bigg[\frac{a\Gamma(2j+3m+1)}{[(i+1)a+t_1+t_2]^{2j+3m+1}}+\frac{b\Gamma(2j+3m+2)}{[(i+1)a+t_1+t_2]^{2j+3m+2}}+\frac{c\Gamma(2j+3m+3)}{[(i+1)a+t_1+t_2]^{2j+3m+3}}\Bigg].$$

Substituting from Equation 40-42 into Equation 37, we have Equation 36, which completes the proof.

Remark 13

The joint moment generating function $M_{X1,X2}(t_1,t_2)$ can be used to derive the expectation of X_1X_2 as:

$$E[X_1X_2]=\left[\frac{\partial^2}{\partial t_1\partial t_2}M_{X_1,X_2}(t_1,t_2)\right]_{t_1=t_2=0}.$$

Maximum Likelihood Estimators

Suppose that $((x_{11},x_{21}),(x_{12},x_{22}),\ldots,(x_{1n},x_{2n}))$ is a random sample from BGQHRD with parameters α_1, α_2, α_3, a, b, c. Consider that:

$$n_1=(\mathrm{i};x_{1\mathrm{i}}<x_{2\mathrm{i}}),\ n_2=(\mathrm{i};x_{1i}>x_{2i}),$$
$$n_3=(\mathrm{i};x_{1i}=x_{2i}=x_i),\ n=n_1+n_2+n_3.$$

For the sample of size n, the likelihood function is given by:

$$\ell(\Phi)=\prod_{i=1}^{n_1}f_1(x_{1i},x_{2i})\prod_{i=1}^{n_2}f_2(x_{1i},x_{2i})\prod_{i=1}^{n_3}f_3(x_i,x_i),$$

where, $\Phi=(\alpha_1,\alpha_2,\alpha_3,a,b,c)$.

Substituting from Equation 9-11 the likelihood function becomes:

$$\ell(\Phi)=(\alpha_1+\alpha_3)^{n_1}(\alpha_2+\alpha_3)^{n_2}\alpha_1^{n_2}\alpha_2^{n_1}\alpha_3^{n_3}\prod_{i=1}^{n_1}(a+bx_{1i}+cx_{1i}^2)$$
$$e^{-\sum_{i=1}^{n_1}\left(ax_{1i}+\frac{b}{2}x_{1i}^2+\frac{c}{3}x_{1i}^3\right)}\prod_{i=1}^{n_1}\left[1-e^{-\left(ax_{1i}+\frac{b}{2}x_{1i}^2+\frac{c}{3}x_{1i}^3\right)}\right]^{\alpha_1+\alpha_3-1}$$
$$\prod_{i=1}^{n_1}\left(a+bx_{2i}+cx_{2i}^2\right)e^{-\sum_{i=1}^{n_1}\left(ax_{2i}+\frac{b}{2}x_{2i}^2+\frac{c}{3}x_{2i}^3\right)}\prod_{i=1}^{n_1}\left[1-e^{-\left(ax_{2}i+\frac{b}{2}x_{2i}^2+\frac{c}{3}x_{2i}^3\right)}\right]^{\alpha_2-1}$$
$$\prod_{i=1}^{n_2}\left(a+bx_{1i}+cx_{1i}^2\right)e^{-\sum_{i=1}^{n_2}\left(ax_{1i}+\frac{b}{2}x_{1i}^2+\frac{c}{3}x_{1i}^3\right)}\times\prod_{i=1}^{n_2}\left[1-e^{-\left(ax_{1i}+\frac{b}{2}x_{1i}^2+\frac{c}{3}x_{1i}^3\right)}\right]^{\alpha_1-1}$$
$$\prod_{i=1}^{n_2}\left(a+bx_{2i}+cx_{2i}^2\right)e^{-\sum_{i=1}^{n_2}\left(ax_{2i}+\frac{b}{2}x_{2i}^2+\frac{c}{3}x_{2i}^3\right)}\prod_{i=1}^{n_2}\left[1-e^{-\left(ax_{2i}+\frac{b}{2}x_{2i}^2+\frac{c}{3}x_{2i}^3\right)}\right]^{\alpha_2+\alpha_3-1}$$
$$\prod_{i=1}^{n_3}\left(a+bx_i+cx_i^2\right)e^{-\sum_{i=1}^{n_3}\left(ax_i+\frac{b}{2}x_i^2+\frac{c}{3}x_i^3\right)}\prod_{i=1}^{n_3}\left[1-e^{-\left(ax_i+\frac{b}{2}x_i^2+\frac{c}{3}x_i^3\right)}\right]^{\alpha_1+\alpha_2+\alpha_3-1}.$$

The log-likelihood function is given as:

$$\begin{aligned}
\mathcal{L}(\Phi)&=n_1\ln(\alpha_1+\alpha_3)+n_1 ln(\alpha_2)+n_2\ln(\alpha_2+\alpha_3)+n_2\ln(\alpha_1)\\
&+n_3\ln(\alpha_3)+\sum_{i=1}^{n_1}\ln\left(a+bx_{1i}+cx_{1i}^2\right)-\sum_{i=1}^{n_1}\left(ax_{1i}+\frac{b}{2}x_{1i}^2+\frac{c}{3}x_{1i}^3\right)\\
&+(\alpha_1+\alpha_3-1)\sum_{i=1}^{n_1}\ln\left[1-e^{-\left(ax_{1i}+\frac{b}{2}x_{1i}^2+\frac{c}{3}x_{1i}^3\right)}\right]+\sum_{i=1}^{n_1}\ln\left(a+bx_{2i}+cx_{2i}^2\right)\\
&-\sum_{i=1}^{n_1}\left(ax_{2i}+\frac{b}{2}x_{2i}^2+\frac{c}{3}x_{2i}^3\right)+(\alpha_2-1)\sum_{i=1}^{n_1}\ln\left[1-e^{-\left(ax_{2i}+\frac{b}{2}x_{2i}^2+\frac{c}{3}x_{2i}^3\right)}\right]\\
&+\sum_{i=1}^{n_2}\ln\left(a+bx_{1i}+cx_{1i}^2\right)-\sum_{i=1}^{n_2}\left(ax_{1i}+\frac{b}{2}x_{1i}^2+\frac{c}{3}x_{1i}^3\right)+(\alpha_1-1)\\
&\times\sum_{i=1}^{n_2}\ln\left[1-e^{-\left(ax_{1i}+\frac{b}{2}x_{1i}^2+\frac{c}{3}x_{1i}^3\right)}\right]+\sum_{i=1}^{n_2}\ln\left(a+bx_{2i}+cx_{2i}^2\right)\\
&-\sum_{i=1}^{n_2}\left(ax_{2i}+\frac{b}{2}x_{2i}^2+\frac{c}{3}x_{2i}^3\right)+(\alpha_2+\alpha_3-1)\\
&\times\sum_{i=1}^{n_2}\ln\left[1-e^{-\left(ax_{2i}+\frac{b}{2}x_{2i}^2+\frac{c}{3}x_{2i}^3\right)}\right]+\sum_{i=1}^{n_3}\ln\left(a+bx_i+cx_i^2\right)\\
&-\sum_{i=1}^{n_3}\left(ax_i+\frac{b}{2}x_i^2+\frac{c}{3}x_i^3\right)+(\alpha_1+\alpha_2+\alpha_3-1)\\
&\times\sum_{i=1}^{n_3}\ln\left[1-e^{-\left(ax_i+\frac{b}{2}x_i^2+\frac{c}{3}x_i^3\right)}\right].
\end{aligned}\tag{43}$$

The first partial derivatives of Equation 43 with respect to α_1, α_2, α_3, a, b and c, can be computed as follows:

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\alpha_1}&=\frac{n_1}{\alpha_1+\alpha_3}+\frac{n_2}{\alpha_1}+\sum_{i=1}^{n_1}\ln\left[1-e^{-\left(ax_{1i}+\frac{b}{2}x_{1i}^2+\frac{c}{3}x_{1i}^3\right)}\right]\\
&+\sum_{i=1}^{n_2}\ln\left[1-e^{-\left(ax_{1i}+\frac{b}{2}x_{1i}^2+\frac{c}{3}x_{1i}^3\right)}\right]+\sum_{i=1}^{n_3}\ln\left[1-e^{-\left(ax_i+\frac{b}{2}x_i^2+\frac{c}{3}x_i^3\right)}\right],
\end{aligned}\tag{44}$$

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\alpha_2}&=\frac{n_2}{\alpha_2+\alpha_3}+\frac{n_1}{\alpha_2}+\sum_{i=1}^{n_1}\ln\left[1-e^{-\left(ax_{2i}+\frac{b}{2}x_{2i}^2+\frac{c}{3}x_{2i}^3\right)}\right]\\
&+\sum_{i=1}^{n_2}\ln\left[1-e^{-\left(ax_{2i}+\frac{b}{2}x_{2i}^2+\frac{c}{3}x_{2i}^3\right)}\right]+\sum_{i=1}^{n_3}\ln\left[1-e^{-\left(ax_i+\frac{b}{2}x_i^2+\frac{c}{3}x_i^3\right)}\right],
\end{aligned}\tag{45}$$

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\alpha_3}&=\frac{n_1}{\alpha_1+\alpha_3}+\frac{n_2}{\alpha_2+\alpha_3}+\frac{n_3}{\alpha_3}+\sum_{i=1}^{n_1}\ln\left[1-e^{-\left(ax_{1i}+\frac{b}{2}x_{1i}^2+\frac{c}{3}x_{1i}^3\right)}\right]\\
&+\sum_{i=1}^{n_2}\ln\left[1-e^{-\left(ax_{2i}+\frac{b}{2}x_{2i}^2+\frac{c}{3}x_{2i}^3\right)}\right]+\sum_{i=1}^{n_3}\ln\left[1-e^{-\left(ax_i+\frac{b}{2}x_i^2+\frac{c}{3}x_i^3\right)}\right],
\end{aligned}\tag{46}$$

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial a}&=\sum_{i=1}^{n_1}\mathcal{A}(x_{1i})-\sum_{i=1}^{n_1}x_{1i}+(\alpha_1+\alpha_3-1)\sum_{i=1}^{n_1}x_{1i}\mathcal{B}(x_{1i})\\
&+\sum_{i=1}^{n_1}\mathcal{A}(x_{2i})-\sum_{i=1}^{n_1}x_{2i}+(\alpha_2-1)\sum_{i=1}^{n_1}x_{2i}\mathcal{B}(x_{2i})+\sum_{i=1}^{n_2}\mathcal{A}(x_{1i})\\
&-\sum_{i=1}^{n_2}x_{1i}+(\alpha_1-1)\sum_{i=1}^{n_2}x_{1i}\mathcal{B}(x_{1i})+\sum_{i=1}^{n_2}\mathcal{A}(x_{2i})-\sum_{i=1}^{n_2}x_{2i}\\
&+(\alpha_2+\alpha_3-1)\sum_{i=1}^{n_2}x_{2i}\mathcal{B}(x_{2i})+\sum_{i=1}^{n_3}\mathcal{A}(x_i)-\sum_{i=1}^{n_3}x_i\\
&+(\alpha_1+\alpha_2+\alpha_3-1)\sum_{i=1}^{n_3}x_i\mathcal{B}(x_i),
\end{aligned}\tag{47}$$

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial b}&=\sum_{i=1}^{n_1}x_{1i}\mathcal{A}(x_{1i})-\frac{1}{2}\sum_{i=1}^{n_1}x_{1i}^2+\frac{1}{2}(\alpha_1+\alpha_3-1)\sum_{i=1}^{n_1}x_{1i}^2\mathcal{B}(x_{1i})\\
&+\sum_{i=1}^{n_1}x_{2i}\mathcal{A}(x_{2i})-\frac{1}{2}\sum_{i=1}^{n_1}x_{2i}^2+\frac{1}{2}(\alpha_2-1)\sum_{i=1}^{n_1}x_{2i}^2\mathcal{B}(x_{2i})\\
&+\sum_{i=1}^{n_2}x_{1i}\mathcal{A}(x_{1i})-\frac{1}{2}\sum_{i=1}^{n_2}x_{1i}^2+\frac{1}{2}(\alpha_1-1)\sum_{i=1}^{n_2}x_{1i}^2\mathcal{B}(x_{1i})\\
&+\sum_{i=1}^{n_2}x_{2i}\mathcal{A}(x_{2i})-\frac{1}{2}\sum_{i=1}^{n_2}x_{2i}^2+\frac{1}{2}(\alpha_2+\alpha_3-1)\sum_{i=1}^{n_2}x_{2i}^2\mathcal{B}(x_{2i})\\
&+\sum_{i=1}^{n_3}x_i\mathcal{A}(x_i)-\frac{1}{2}\sum_{i=1}^{n_3}x_i^2+\frac{1}{2}(\alpha_1+\alpha_2+\alpha_3-1)\sum_{i=1}^{n_3}x_i^2\mathcal{A}(x_i),
\end{aligned}\tag{48}$$

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial c}&=\sum_{i=1}^{n_1}x_{1i}^2\mathcal{A}(x_{1i})-\frac{1}{3}\sum_{i=1}^{n_1}x_{1i}^3+\frac{1}{3}(\alpha_1+\alpha_3-1)\sum_{i=1}^{n_1}x_{1i}^3\mathcal{B}(x_{1i})\\
&+\sum_{i=1}^{n_1}x_{2i}^2\mathcal{A}(x_{2i})-\frac{1}{3}\sum_{i=1}^{n_1}x_{2i}^3+\frac{1}{3}(\alpha_2-1)\sum_{i=1}^{n_1}x_{2i}^3\mathcal{B}(x_{2i})\\
&+\sum_{i=1}^{n_2}x_{1i}^2\mathcal{A}(x_{1i})-\frac{1}{3}\sum_{i=1}^{n_2}x_{1i}^3+\frac{1}{3}(\alpha_1-1)\sum_{i=1}^{n_2}x_{1i}^3\mathcal{B}(x_{1i})\\
&+\sum_{i=1}^{n_2}x_{2i}^2\mathcal{A}(x_{2i})-\frac{1}{3}\sum_{i=1}^{n_2}x_{2i}^3+\frac{1}{3}(\alpha_2+\alpha_3-1)\sum_{i=1}^{n_2}x_{2i}^3\mathcal{B}(x_{2i})\\
&+\sum_{i=1}^{n_3}x_i^2\mathcal{A}(x_i)-\frac{1}{3}\sum_{i=1}^{n_3}x_i^3+\frac{1}{3}(\alpha_1+\alpha_2+\alpha_3-1)\sum_{i=1}^{n_3}x_i^3\mathcal{B}(x_i),
\end{aligned}\tag{49}$$

where:

$$\mathcal{A}(x)=\frac{1}{a+bx+cx^2},\quad \mathcal{B}(x)=\frac{e^{-\left(ax+\frac{b}{2}x^2+\frac{c}{3}x^3\right)}}{1-e^{-\left(ax+\frac{b}{2}x^2+\frac{c}{3}x^3\right)}}.$$

To get the MLEs of the parameters α_1, α_2, α_3, a, b and c, we have to solve the above system on nonlinear equation with respect to α_1, α_2, α_3, a, b and c. The solution of the Equation 44 to 49 are not easy to solve. Therefore a numerical technique can be used to get the MLEs.

Based on the asymptotic distributions of the MLEs the approximate confidence intervals of the parameters are derived. We determine the second partial derivatives to obtain the information matrix for α_1, α_2, α_3, a, b and c as follows:

$$\frac{\partial^2\mathcal{L}}{\partial\alpha_1^2} = -\frac{n_1}{(\alpha_1+\alpha_3)^2} - \frac{n_2}{\alpha_1^2}, \quad (50)$$

$$\frac{\partial^2\mathcal{L}}{\partial\alpha_1\partial\alpha_2} = 0, \quad (51)$$

$$\frac{\partial^2\mathcal{L}}{\partial\alpha_1\partial\alpha_3} = -\frac{n_1}{(\alpha_1+\alpha_3)^2}, \quad (52)$$

$$\frac{\partial^2\mathcal{L}}{\partial\alpha_1\partial a} = \sum_{i=1}^{n_1} x_{1i}\mathcal{B}(x_{1i}) + \sum_{i=1}^{n_2} x_{1i}\mathcal{B}(x_{1i}) + \sum_{i=1}^{n_3} x_i\mathcal{B}(x_i), \quad (53)$$

$$\frac{\partial^2\mathcal{L}}{\partial\alpha_1\partial b} = \frac{1}{2}\left[\sum_{i=1}^{n_1} x_{1i}^2\mathcal{B}(x_{1i}) + \sum_{i=1}^{n_2} x_{1i}^2\mathcal{B}(x_{1i}) + \sum_{i=1}^{n_3} x_i^2\mathcal{B}(x_i)\right], \quad (54)$$

$$\frac{\partial^2\mathcal{L}}{\partial\alpha_1\partial c} = \frac{1}{3}\left[\sum_{i=1}^{n_1} x_{1i}^3\mathcal{B}(x_{1i}) + \sum_{i=1}^{n_2} x_{1i}^3\mathcal{B}(x_{1i}) + \sum_{i=1}^{n_3} x_i^3\mathcal{B}(x_i)\right], \quad (55)$$

$$\frac{\partial^2\mathcal{L}}{\partial\alpha_2^2} = -\frac{n_2}{(\alpha_2+\alpha_3)^2} - \frac{n_1}{\alpha_2^2}, \quad (56)$$

$$\frac{\partial^2\mathcal{L}}{\partial\alpha_2\partial\alpha_3} = -\frac{n_2}{(\alpha_2+\alpha_3)^2}, \quad (57)$$

$$\frac{\partial^2\mathcal{L}}{\partial\alpha_2\partial a} = \sum_{i=1}^{n_1} x_{2i}\mathcal{B}(x_{2i}) + \sum_{i=1}^{n_2} x_{2i}\mathcal{B}(x_{2i}) + \sum_{i=1}^{n_3} x_i\mathcal{B}(x_i), \quad (58)$$

$$\frac{\partial^2\mathcal{L}}{\partial\alpha_2\partial b} = \frac{1}{2}\left[\sum_{i=1}^{n_1} x_{2i}^2\mathcal{B}(x_{2i}) + \sum_{i=1}^{n_2} x_{2i}^2\mathcal{B}(x_{2i}) + \sum_{i=1}^{n_3} x_i^2\mathcal{B}(x_i)\right], \quad (59)$$

$$\frac{\partial^2\mathcal{L}}{\partial\alpha_2\partial c} = \frac{1}{3}\left[\sum_{i=1}^{n_1} x_{2i}^3\mathcal{B}(x_{2i}) + \sum_{i=1}^{n_2} x_{2i}^3\mathcal{B}(x_{2i}) + \sum_{i=1}^{n_3} x_i^3\mathcal{B}(x_i)\right], \quad (60)$$

$$\frac{\partial^2\mathcal{L}}{\partial\alpha_3^2} = -\frac{n_1}{(\alpha_1+\alpha_3)^2} - \frac{n_2}{(\alpha_2+\alpha_3)^2} - \frac{n_3}{\alpha_3^2} \quad (61)$$

$$\frac{\partial^2\mathcal{L}}{\partial\alpha_3\partial a} = \sum_{i=1}^{n_1} x_{1i}\mathcal{B}(x_{1i}) + \sum_{i=1}^{n_2} x_{2i}\mathcal{B}(x_{2i}) + \sum_{i=1}^{n_3} x_i\mathcal{B}(x_i), \quad (62)$$

$$\frac{\partial^2\mathcal{L}}{\partial\alpha_3\partial b} = \frac{1}{2}\left[\sum_{i=1}^{n_1} x_{1i}^2\mathcal{B}(x_{1i}) + \sum_{i=1}^{n_2} x_{2i}^2\mathcal{B}(x_{2i}) + \sum_{i=1}^{n_3} x_i^2\mathcal{B}(x_i)\right], \quad (63)$$

$$\frac{\partial^2\mathcal{L}}{\partial\alpha_3\partial c} = \frac{1}{3}\left[\sum_{i=1}^{n_1} x_{1i}^3\mathcal{B}(x_{1i}) + \sum_{i=1}^{n_2} x_{2i}^3\mathcal{B}(x_{2i}) + \sum_{i=1}^{n_3} x_i^3\mathcal{B}(x_i)\right], \quad (64)$$

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial a^2} = &-\sum_{i=1}^{n_1}\mathcal{A}^2(x_{1i}) - (\alpha_1+\alpha_3-1)\sum_{i=1}^{n_1}x_{1i}^2\mathcal{C}(x_{1i}) - \sum_{i=1}^{n_1}\mathcal{A}^2(x_{2i})\\ &-(\alpha_2-1)\sum_{i=1}^{n_1}x_{2i}^2\mathcal{C}(x_{2i}) - \sum_{i=1}^{n_2}\mathcal{A}^2(x_{1i}) - (\alpha_1-1)\sum_{i=1}^{n_2}x_{1i}^2\mathcal{C}(x_{1i})\\ &-\sum_{i=1}^{n_2}\mathcal{A}^2(x_{2i}) - (\alpha_2+\alpha_3-1)\sum_{i=1}^{n_2}x_{2i}^2\mathcal{C}(x_{2i}) - \sum_{i=1}^{n_3}\mathcal{A}^2(x_i)\\ &-(\alpha_1+\alpha_2+\alpha_3-1)\sum_{i=1}^{n_3}x_i^2\mathcal{C}(x_i),\end{aligned} \quad (65)$$

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial a\partial b} = &-\sum_{i=1}^{n_1}x_{i1}\mathcal{A}^2(x_{1i}) - \frac{1}{2}(\alpha_1+\alpha_3-1)\sum_{i=1}^{n_1}x_{1i}^3\mathcal{C}(x_{1i})\\ &-\sum_{i=1}^{n_1}x_{2i}\mathcal{A}^2(x_{2i}) - \frac{1}{2}(\alpha_2-1)\sum_{i=1}^{n_1}x_{2i}^3\mathcal{C}(x_{2i}) - \sum_{i=1}^{n_2}x_{1i}\mathcal{A}^2(x_{1i})\\ &-\frac{1}{2}(\alpha_1-1)\sum_{i=1}^{n_2}x_{1i}^3\mathcal{C}(x_{1i}) - \sum_{i=1}^{n_2}x_{2i}\mathcal{A}^2(x_{2i}) - \frac{1}{2}(\alpha_2+\alpha_3-1)\\ &\times\sum_{i=1}^{n_2}x_{2i}^3\mathcal{C}(x_{2i}) - \sum_{i=1}^{n_3}x_i\mathcal{A}^2(x_i) - \frac{1}{2}(\alpha_1+\alpha_2+\alpha_3-1)\sum_{i=1}^{n_3}x_i^3\mathcal{C}(x_i),\end{aligned} \quad (66)$$

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial a\partial c} = &-\sum_{i=1}^{n_1}x_{i1}^2\mathcal{A}^2(x_{1i}) - \frac{1}{3}(\alpha_1+\alpha_3-1)\sum_{i=1}^{n_1}x_{1i}^4\mathcal{C}(x_{1i})\\ &-\sum_{i=1}^{n_1}x_{2i}^2\mathcal{A}^2(x_{2i}) - \frac{1}{3}(\alpha_2-1)\sum_{i=1}^{n_1}x_{2i}^4\mathcal{C}(x_{2i}) - \sum_{i=1}^{n_2}x_{1i}^2\mathcal{A}^2(x_{1i})\\ &-\frac{1}{3}(\alpha_1-1)\sum_{i=1}^{n_2}x_{1i}^4\mathcal{C}(x_{1i}) - \sum_{i=1}^{n_2}x_{2i}^2\mathcal{A}^2(x_{2i}) - \frac{1}{3}(\alpha_2+\alpha_3-1)\\ &\times\sum_{i=1}^{n_2}x_{2i}^4\mathcal{C}(x_{2i}) - \sum_{i=1}^{n_3}x_i^2\mathcal{A}^2(x_i) - \frac{1}{3}(\alpha_1+\alpha_2+\alpha_3-1)\sum_{i=1}^{n_3}x_i^4\mathcal{C}(x_i),\end{aligned} \quad (67)$$

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial b^2} = &-\sum_{i=1}^{n_1}x_{i1}^2\mathcal{A}^2(x_{1i}) - \frac{1}{4}(\alpha_1+\alpha_3-1)\sum_{i=1}^{n_1}x_{1i}^4\mathcal{C}(x_{1i})\\ &-\sum_{i=1}^{n_1}x_{2i}^2\mathcal{A}^2(x_{2i}) - \frac{1}{4}(\alpha_2-1)\sum_{i=1}^{n_1}x_{2i}^4\mathcal{C}(x_{2i}) - \sum_{i=1}^{n_2}x_{1i}^2\mathcal{A}^2(x_{1i})\\ &-\frac{1}{4}(\alpha_1-1)\sum_{i=1}^{n_2}x_{1i}^4\mathcal{C}(x_{1i}) - \sum_{i=1}^{n_2}x_{2i}^2\mathcal{A}^2(x_{2i}) - \frac{1}{4}(\alpha_2+\alpha_3-1)\\ &\times\sum_{i=1}^{n_2}x_{2i}^4\mathcal{C}(x_{2i}) - \sum_{i=1}^{n_3}x_i^2\mathcal{A}^2(x_i) - \frac{1}{4}(\alpha_1+\alpha_2+\alpha_3-1)\sum_{i=1}^{n_3}x_i^4\mathcal{C}(x_i),\end{aligned} \quad (68)$$

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial b\partial c} = &-\sum_{i=1}^{n_1}x_{1i}^3\mathcal{A}^2(x_{1i}) - \frac{1}{6}(\alpha_1+\alpha_3-1)\sum_{i=1}^{n_1}x_{1i}^5\mathcal{C}(x_{1i})\\ &-\sum_{i=1}^{n_1}x_{2i}^3\mathcal{A}^2(x_{2i}) - \frac{1}{6}(\alpha_2-1)\sum_{i=1}^{n_1}x_{2i}^5\mathcal{C}(x_{2i}) - \sum_{i=1}^{n_2}x_{1i}^3\mathcal{A}^2(x_{1i})\\ &-\frac{1}{6}(\alpha_1-1)\sum_{i=1}^{n_2}x_{1i}^5\mathcal{C}(x_{1i}) - \sum_{i=1}^{n_2}x_{2i}^3\mathcal{A}^2(x_{2i}) - \frac{1}{6}(\alpha_2+\alpha_3-1)\\ &\times\sum_{i=1}^{n_2}x_{2i}^5\mathcal{C}(x_{2i}) - \sum_{i=1}^{n_3}x_i^3x_i\mathcal{A}^2(x_i) - \frac{1}{6}(\alpha_1+\alpha_2+\alpha_3-1)\sum_{i=1}^{n_3}x_i^5\mathcal{C}(x_i),\end{aligned} \quad (69)$$

$$\frac{\partial\mathcal{L}}{\partial c^2} = -\sum_{i=1}^{n_1} x_{i1}^4\mathcal{A}^2(x_{1i}) - \frac{1}{9}(\alpha_1+\alpha_3-1)\sum_{i=1}^{n_1} x_{1i}^6\mathcal{C}(x_{1i})$$
$$-\sum_{i=1}^{n_1} x_{2i}^4\mathcal{A}^2(x_{2i}) - \frac{1}{9}(\alpha_2-1)\sum_{i=1}^{n_1} x_{2i}^6\mathcal{C}(x_{2i}) - \sum_{i=1}^{n_2} x_{1i}^4\mathcal{A}^2(x_{1i}) \quad (70)$$
$$-\frac{1}{9}(\alpha_1-1)\sum_{i=1}^{n_2} x_{1i}^6\mathcal{C}(x_{1i}) - \sum_{i=1}^{n_2} x_{2i}^4\mathcal{A}^2(x_{2i}) - \frac{1}{9}(\alpha_2+\alpha_3-1)$$
$$\times\sum_{i=1}^{n_2} x_{2i}^6\mathcal{C}(x_{2i}) - \sum_{i=1}^{n_3} x_i^4\mathcal{A}^2(x_i) - \frac{1}{9}(\alpha_1+\alpha_2+\alpha_3-1)\sum_{i=1}^{n_3} x_i^6\mathcal{C}(x_i),$$

where:

$$\mathcal{C}(x) = \frac{e^{-\left(ax+\frac{b}{2}x^2+\frac{c}{3}x^3\right)}}{\left[1-e^{-\left(ax+\frac{b}{2}x^2+\frac{c}{3}x^3\right)}\right]^2}.$$

Then the variance-covariance matrix may be approximated by:

$$V = I^{-1} = \begin{pmatrix} I_{11} & I_{12} & I_{13} & I_{14} & I_{15} & I_{16} \\ I_{21} & I_{22} & I_{23} & I_{24} & I_{25} & I_{26} \\ I_{31} & I_{32} & I_{33} & I_{34} & I_{35} & I_{36} \\ I_{41} & I_{42} & I_{43} & I_{44} & I_{45} & I_{46} \\ I_{51} & I_{52} & I_{53} & I_{54} & I_{55} & I_{56} \\ I_{61} & I_{62} & I_{63} & I_{64} & I_{65} & I_{66} \end{pmatrix}^{-1}, \quad (71)$$

where, I is the observed information matrix and:

$$I_{ij} = E\left[-\frac{\partial^2 L}{\partial\theta\partial\theta^`}\right], \quad \text{where } \theta,\theta^` = \alpha_1,\alpha_2,\alpha_3,a,b,c.$$

Using Equation 71, a $(1-\gamma)100\%$ confidence interval for α_1, α_2, α_3, a, b and c are approximated respectively as:

$$\alpha_i \pm z_{\frac{\gamma}{2}}\sqrt{V_{\alpha_i\alpha_i}}, \quad i=1,2,3, \qquad a \pm z_{\frac{\gamma}{2}}\sqrt{V_{aa}},$$
$$b \pm z_{\frac{\gamma}{2}}\sqrt{V_{bb}}, \qquad c \pm z_{\frac{\gamma}{2}}\sqrt{V_{cc}},$$

where, $z_{\gamma/2}$ is the upper $(\gamma/2)th$ percentile of the standard normal distribution.

Data Analysis

The following data represent the American Football (National Football League) League data and they are obtained from the matches played on three consecutive weekends in 1986.

The data (scoring times in minutes and seconds) are represented in the following Table 1.

From this data, we find that the values of $\hat{\alpha}_i, i=1,2,3,\ \hat{a},\ \hat{b}$ and $\hat{c}$ are 0.05, 0.75, 1.547, 0.151, -5.842×10^{-3} and 1.065×10^{-4}, respectively and the log-likelihood equals (-239.318).

By substituting the MLEs of these parameters in Equation 71, we obtain the estimation of the variance covariance matrix as:

$$\begin{pmatrix} 1.272\times10^{-3} & 1.0362\times10^{-3} & 1.6923\times10^{-3} & 2.7367\times10^{-4} & -2.4462\times10^{-5} & 4.3822\times10^{-7} \\ 1.0362\times10^{-3} & 0.0725 & 0.0784 & 9.9959\times10^{-3} & -8.8618\times10^{-4} & 1.5847\times10^{-5} \\ 1.6923\times10^{-3} & 0.0784 & 0.2262 & 0.021 & -1.883\times10^{-3} & 3.3742\times10^{-5} \\ 2.7367\times10^{-4} & 9.9959\times10^{-3} & 0.021 & 2.722\times10^{-3} & -2.5713\times10^{-4} & 4.723\times10^{-6} \\ -2.4462\times10^{-5} & -8.8618\times10^{-4} & -1.883\times10^{-3} & -2.5713\times10^{-4} & 2.7573\times10^{-5} & -5.37\times10^{-7} \\ 4.3822\times10^{-7} & 1.5847\times10^{-5} & 3.3742\times10^{-5} & 4.723\times10^{-6} & -5.37\times10^{-7} & 1.114\times10^{-8} \end{pmatrix}.$$

Table 1. American Football (National Football League) League data

X_1	X_2	X_1	X_2	X_1	X_2
2.05	3.98	5.78	25.98	10.40	10.25
9.05	9.05	13.8	49.75	2.98	2.98
0.85	0.85	7.25	7.25	3.88	6.43
3.43	3.43	4.25	4.25	0.75	0.75
7.78	7.78	1.65	1.65	11.63	17.37
10.57	14.28	6.42	15.08	1.38	1.38
7.05	7.05	4.22	9.48	10.53	10.53
2.58	2.58	15.53	15.53	12.13	12.13
7.23	9.68	2.90	2.90	14.58	14.58
6.85	34.58	7.02	7.02	11.82	11.82
32.45	42.35	6.42	6.42	5.52	11.27
8.53	14.57	8.98	8.98	19.65	10.70
31.13	49.88	10.15	10.15	17.83	17.83
14.58	20.57	8.87	8.87	10.85	38.07

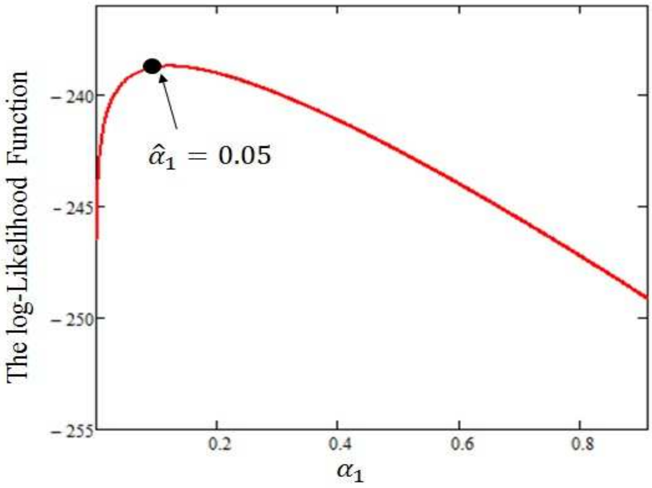

Fig. 1. The profile of the log-likelihood function of α_1

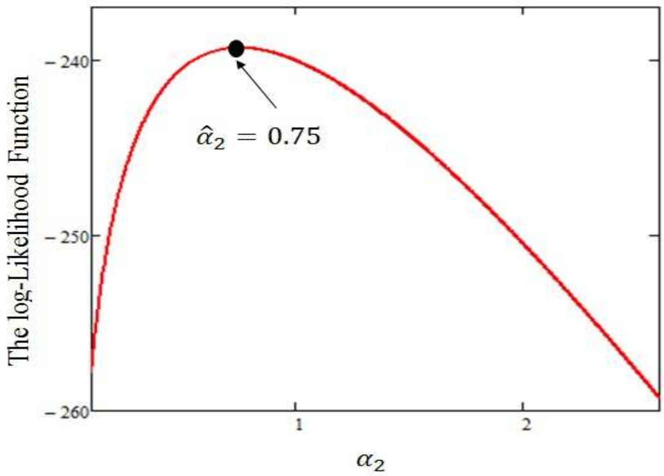

Fig. 2. The profile of the log-likelihood function of α_2

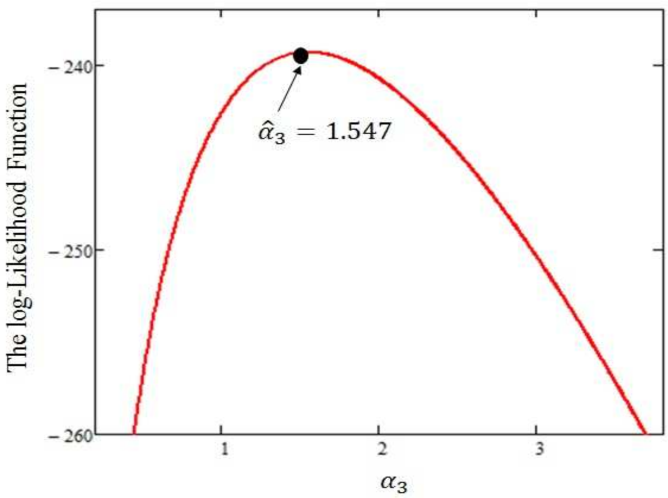

Fig. 3. The profile of the log-likelihood function of α_3

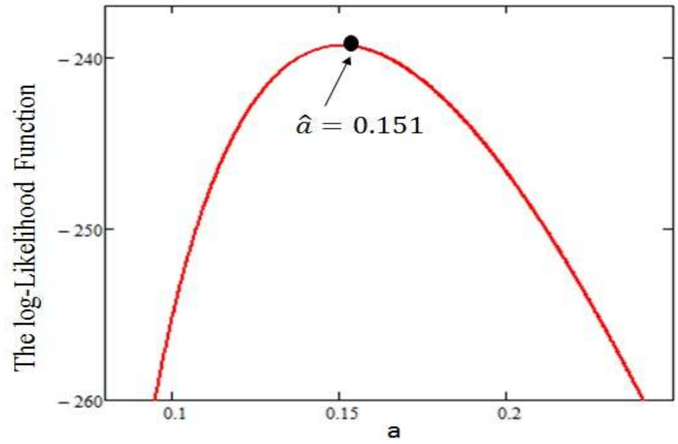

Fig. 4. The profile of the log-likelihood function of a

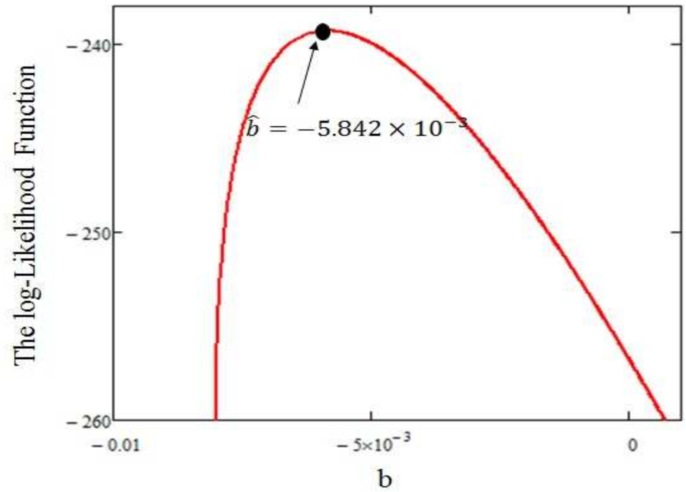

Fig. 5. The profile of the log-likelihood function of b

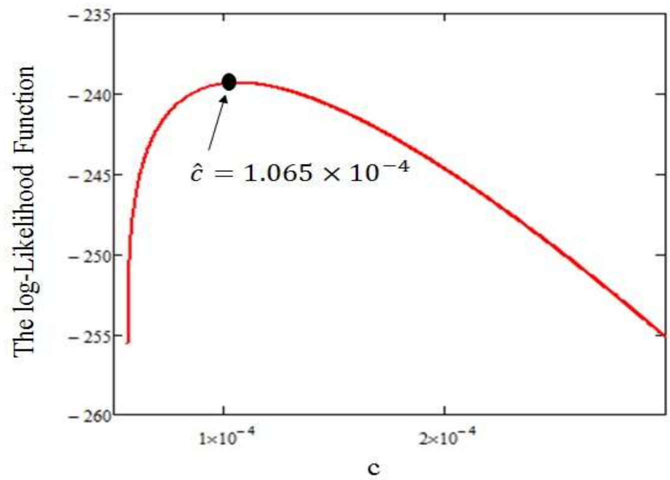

Fig. 6. The profile of the log-likelihood function of c

The approximated 95% confidence interval of α_1, α_2, α_3, a, b and c are given by [0, 0.1199], [0.22237, 1.27763], [0.61491, 2.47909], [0.04874, 0.25326], [-0.01613, 4.45001×10^{-3}] and [0, 3.13368×10^{-4}], respectively.

We plot the profiles of the log-likelihood function of α_1, α_2, α_3, a, b and c, respectively, Fig. 1-6.

Conclusion

In this study, we investigate the BGQHRD. Also, we determine the joint probability density function, the joint cumulative distribution function and joint survival distribution function. Moreover, several properties of this distribution have been discussed. Conditional probability density functions, *r*th moments and the joint and marginal moment generating functions are derived. Parameters estimators using the maximum likelihood method are discussed. A numerical illustration by using real data is used to obtain Maximum Likelihood Estimators (MLEs) and the behavior of the estimators numerically is studied.

Ethics

The author declares that there is no conflict of interests regarding the publication of this article.

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