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Bivariate Poisson-Lindley Distribution with Application

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Abstract: This study applies a Bivariate Poisson-Lindley (BPL) distribution for modeling dependent and over-dispersed count data. The advantage of using this form of BPL distribution is that the correlation coefficient can be positive, zero or negative, depending on the multiplicative factor parameter. Several properties such as mean, variance and correlation coefficient of the BPL distribution are discussed. A numerical example is given and the BPL distribution is compared to Bivariate Poisson (BP) and Bivariate Negative Binomial (BNB) distributions which also allow the correlation coefficient to be positive, zero or negative. The results show that BPL distribution provides the smallest Akaike Information Criterion (AIC), indicating that the distribution can be used as an alternative for fitting dependent and over-dispersed count data, with either negative or positive correlation.

Keywords: Bivariate, Poisson-Lindley, Dependent, Over-Dispersed, Count Data

Introduction

Mixed Poisson and mixed negative binomial distributions have been considered as alternatives for fitting count data with overdispersion. Examples of mixed Poisson and mixed negative binomial distributions are Negative Binomial (NB) obtained as a mixture of Poisson and gamma, Poisson-Lindley (PL) (Sankaran, 1970; Ghitany *et al*., 2008), Poisson-Inverse Gaussian (PIG) (Trembley, 1992; Willmot, 1987), Negative Binomial-Pareto (NBP) (Meng *et al*., 1999), Negative Binomial-Inverse Gaussian (NBIG) (Gomez-Deniz *et al*., 2008), negative binomial-Lindley (NBL) (Zamani and Ismail, 2010; Lord and Geedipally, 2011) and Poisson-Weighted Exponential (PWE) (Zamani *et al*., 2014a) distributions.

Based on literatures, the mixture approaches have been used to derive new families of bivariate distribution. The Bivariate Negative Binomial (BNB), Bivariate Poisson-Lognormal (BPLN), Bivariate Poisson-Inverse Gaussian (BPIG) and bivariate Poisson-Lindley (BPL) distributions are several examples of classes of mixed distribution which are extended from univariate case. For further literatures, BNB distribution was studied in Marshall and Olkin (1990) and applied in Karlis and Ntzoufras (2003), tests for overdispersion and independence in BNB model were discussed in (Jung *et al*., 2009; Cheon *et al*., 2009), BPIG distribution was derived by Stein *et al*. (1987), BPL distribution was proposed by Gomez-Deniz *et al*. (2012) and Bivariate Poisson-Weighted Exponential (BPWE) was proposed in Zamani *et al*. (2014b).

Besides mixture approach, several bivariate discrete distributions have been defined using the method of trivariate reduction (Kocherlakota and Kocherlakota, 1999; Johnson *et al*., 1997). The BP distribution from the trivariate reduction has been used for modeling correlated bivariate count data and several applications can be found in (Holgate, 1964; Paul and Ho, 1989). Besides BP distribution, the Bivariate Generalized Poisson (BGP) distribution from the trivariate reduction has been defined and studied in Famoye and Consul (1995), where the distribution can be used for modeling correlated and under- or overdispersed bivariate count data.

In this study, we apply the BPL distribution which was derived from the product of two PL marginals with a multiplicative factor parameter. This BPL distribution can be used for bivariate count data with positive, zero or negative correlation coefficient. The rest of this study is organized as follows. Section 2 provides the univariate version of PL distribution. Several properties of the BPL distribution, such as mean, variance and correlation coefficient, are discussed in section 3. Section 4 discusses parameter estimation for the BPL and section 5 provides several tests for testing independence. Numerical illustration is provided in section 6, where BPL distribution is fitted to the bivariate flight aborts count data. The BPL distribution is compared to BP (Lakshminarayana *et al*., 1999) and BNB (Famoye, 2010) distributions which also allow positive, zero or negative correlation.

Univariate Poisson-Lindley (PL) Distribution

The Lindley (θ) distribution has the following *p.d.f.* (Lindley, 1958):

$$f(x)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x},\quad x>0,\theta>0$$

And *m.g.f*:

$$M_X(z)=\frac{\theta^2}{\theta+1}\frac{\theta-z+1}{(\theta-z)^2} \tag{1}$$

Assuming the conditional random variable $Y|\Lambda$ follows Poisson distribution with *p.m.f*:

$$\Pr(Y=y\mid\lambda)=\frac{e^{-\lambda\xi}(\lambda\xi)^y}{y!},\quad y=0,1,2,...,\quad \lambda,\mu>0$$

The random variable Λ is distributed as Lindley (θ), the marginal distribution of the random variable Y is distributed as PL (θ,ξ) which is:

$$\Pr(Y=y)=\frac{\theta^2\xi^y}{1+\theta}\frac{\theta+\xi+y+1}{(\theta+\xi)^{y+2}},\ y=0,1,2,...,\ \theta,\xi>0 \tag{2}$$

Bivariate Poisson-Lindley (BPL) Distribution

By setting ξ = 1 in (2), the *p.m.f.* of PL (θ) distribution which is obtained in Sankaran (1970) is:

$$\Pr(Y=y)=\frac{\theta^2(\theta+y+2)}{(\theta+1)^{y+3}},\ y=0,1,2,...,\ \theta>0$$

With mean:

$$E(\mathrm{Y})=\frac{\theta+2}{\theta(\theta+1)}$$

Variance:

$$Var(Y)=\frac{\theta^3+4\theta^2+6\theta+2}{\theta^2(\theta+1)^2}$$

And *m.g.f*:

$$M_Y(z)=M_\Lambda(e^z-1)=E(e^{ZY})$$
$$=\frac{\theta^2}{1+\theta}\frac{\theta+2-e^z}{(\theta-e^z+1)^2} \tag{3}$$

In this study, we use the BPL distribution which was derived by Gomez-Deniz *et al.* (2012), who used the methodology proposed by Lee (1996) and ideas suggested in Sarmanov (1966). The same approach was also used by Lakshminarayana *et al.* (1999) for deriving BP distribution. The joint *p.m.f.* of BP (λ_1, λ_1, α) distribution, which was derived from the product of two Poisson marginals with a multiplicative factor parameter, is defined as (Lakshminarayana *et al.*, 1999):

$$\Pr(Y_1=y_1,Y_2=y_2)=e^{-\lambda_1-\lambda_2}\frac{\lambda_1^{y_1}\lambda_2^{y_2}}{y_1!y_2!}$$
$$\{1+\alpha[(g_1(y_1)-\bar{g}_1)(g_2(y_2)-\bar{g}_2)]\} \tag{4}$$
$$y_1,y_2=0,1,2,...,\quad \lambda_1,\lambda_2>0$$

where, $g_1(y_1)$ and $g_2(y_2)$ are bounded functions in y_1 and y_1 respectively. The value of {.} in (4) is non-negative when $g_t(y_t)=e^{-y_t}$ and $\bar{g}_t=E[g_t(Y_t)]=E(e^{-Y_t}),\ \ t=1,2$.

In a similar manner, the joint *p.m.f.* of BPL (θ_1, θ_2, α) distribution is defined as:

$$P(Y_1=y_1,Y_2=y_2)=\frac{\theta_1^2(y_1+\theta_1+2)}{(\theta_1+1)^{y_1+3}}\frac{\theta_2^2(y_2+\theta_2+2)}{(\theta_2+1)^{y_2+3}}$$
$$[1+\alpha(e^{-y_1}-c_1)(e^{-y_2}-c_2)],\ y_1,y_2=0,1,2,...,\quad \theta_1,\theta_2>0 \tag{5}$$

Where:

$$c_t=E(e^{-Y_t})=\frac{\theta_t^2}{1+\theta_t}\frac{\theta_t+2-e^{-1}}{(\theta_t-e^{-1}+1)^2},\ t=1,2 \tag{6}$$

We obtain $E(e^{-Y_t})$ in (6) by letting $z=-1$ in *m.g.f.* (3). When α = 0, random variables Y_1 and Y_2 are independent, each is distributed as a marginal PL. Therefore, α is the parameter of independence.

The first five moments of BPL (θ_1, θ_2, α) distribution are:

$$E(Y_t)=\mu_t=\frac{\theta_t+2}{\theta_t(\theta_t+1)},\quad t=1,2$$

$$Var(Y_t)=\sigma_t^2=\frac{\theta_t^3+4\theta_t^2+6\theta_t+2}{\theta_t^2(\theta_t+1)^2},\quad t=1,2$$

And:

$$Cov(Y_1,Y_2)=\alpha(c_{11}-\mu_1c_1)(c_{22}-\mu_2c_2) \tag{7}$$

where, $c_{tt}=E(Y_te^{-Y_t}),\ t=1,2$ differentiating *m.g.f.* in (3) with respect to z and letting $z=-1$, we have $\left.\frac{\partial}{\partial z}M_Y(z)\right|_{z=-1}=E(Ye^{-Y})$. Thus, $c_{tt}=\frac{\theta_t^2}{1+\theta_t}\frac{\theta_te^{-1}-e^{-2}+3e^{-1}}{(\theta_t-e^{-1}+1)^3},\ t=1,2$.

Using the variance and covariance in (7), the correlation coefficient is:

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$$\rho_{12} = \frac{\sigma_{12}}{\sigma_1\sigma_2} = \frac{\alpha(c_{11}-\mu_1 c_1)(c_{22}-\mu_2 c_2)}{\sigma_1\sigma_2} \tag{8}$$

From (8), Y_1 and Y_2 are independent when $\alpha = 0$ and have positive and negative correlations when $\alpha>0$ and $\alpha<0$ respectively.

Parameter Estimation

The moment estimates of BPL (θ_1, θ_2, α) distribution can be obtained by equating the mean and covariance in (7) with the sample moments $\bar{y}_t = \frac{\sum_{i=1}^{n} y_{it}}{n}$, $t = 1,2$ and $s_{12} = \frac{\sum_{i=1}^{n}(y_{i1}-\bar{y}_1)(y_{i2}-\bar{y}_2)}{n-1}$.

Following Sankaran (1970), the unique moment estimate of θ_t is $\tilde{\theta}_t = \frac{-(\bar{y}_t - 1) + \sqrt{(\bar{y}_t - 1)^2 + 8\bar{y}_t}}{2\bar{y}_t}$, $\bar{y}_t > 0$, $t = 1,2$. The moment estimate for α can then be computed using $\tilde{\alpha} = s_{12}(\tilde{c}_{11} - \tilde{c}_1\bar{y}_1)^{-1}(\tilde{c}_{22} - \tilde{c}_2\bar{y}_2)^{-1}$, where $\tilde{c}_1, \tilde{c}_2, \tilde{c}_{11}$ and $\tilde{c}_{22}$ are estimated values of c_1, c_2, c_{11} and c_{22}.

The log likelihood function for BPL (θ_1, θ_2, α) distribution is:

$$\begin{aligned}\log L = \sum_{i=1}^{n}\{&2\log\theta_1 + \log(y_{i1}+\theta_1+2) - (y_{i1}+3)\log(\theta_1+1)\\ &+2\log\theta_2 + \log(y_{i2}+\theta_2+2) - (y_{i2}+3)\log(\theta_2+1)\\ &+\log[1+\alpha(e^{-y_{i1}}-c_1)(e^{-y_{i2}}-c_2)]\}.\end{aligned} \tag{9}$$

The log likelihood estimates of BPL (θ_1, θ_2, α) can be obtained by maximizing the log likelihood in (9). The Fisher Information matrix can be obtained using the negative expectation of the second derivatives of log likelihood.

Several Tests

As mentioned previously, when $\alpha = 0$, random variables Y_1 and Y_1 are independent, each is distributed as a marginal PL. For testing independence, we can test H_0: $\alpha = 0$ against H_1: $\alpha \neq 0$ and the test can be performed using Likelihood Ratio Test (LRT), $T = 2(\ln L_1 - \ln L_0)$, where $\ln L_1$ and $\ln L_0$ are the models' log likelihood under their respective hypothesis. The statistic is approximately distributed as a chi-square with one degree of freedom.

The test of independence can also be performed using Wald statistic which is, $\frac{\hat{\alpha}^2}{\hat{V}ar(\hat{\alpha})}$, where $\hat{\alpha}$ is the estimate of independence parameter and $\hat{V}ar(\hat{\alpha})$ is its estimated variance. The Wald statistic is approximately distributed as a chi-square with one degree of freedom. The variance of parameters for BPL distribution can be estimated using the diagonal elements of the inverse of negative Hessian matrix. The elements of Hessian matrix are the second derivatives of log likelihood.

As another alternative, we can also use a score statistic, which is further discussed in Cox and Hinkley (1979). For the score test, we need the score function, $U(\theta_1, \theta_2, \alpha = 0)$ and the expected information matrix, $I(\theta_1, \theta_2, \alpha = 0)$, which can be obtained from the log likelihood.

The score statistic for testing H_0: $\alpha = 0$ against H_1: $\alpha \neq 0$ is:

$$S = U'(\hat{\theta}_1, \hat{\theta}_2, \alpha = 0)[I(\hat{\theta}_1, \hat{\theta}_2, \alpha = 0)]^{-1} U(\hat{\theta}_1, \hat{\theta}_2, \alpha = 0) \tag{10}$$

Where:

$$U(\hat{\theta}_1, \hat{\theta}_2, \alpha = 0) = \left(\frac{\partial\ell}{\partial\theta_1}, \frac{\partial\ell}{\partial\theta_2}, \frac{\partial\ell}{\partial\alpha}\right)\Bigg|_{(\theta_1,\theta_2,0)}$$

And:

$$I(\hat{\theta}_1, \hat{\theta}_2, \alpha = 0) = \left.\begin{pmatrix} -E\left(\frac{\partial^2\ell}{\partial\theta_1^2}\right) & -E\left(\frac{\partial^2\ell}{\partial\theta_1\partial\theta_2}\right) & -E\left(\frac{\partial^2\ell}{\partial\theta_1\partial\alpha}\right) \\ -E\left(\frac{\partial^2\ell}{\partial\theta_2\partial\theta_1}\right) & -E\left(\frac{\partial^2\ell}{\partial\theta_2^2}\right) & -E\left(\frac{\partial^2\ell}{\partial\theta_2\partial\alpha}\right) \\ -E\left(\frac{\partial^2\ell}{\partial\alpha\partial\theta_1}\right) & -E\left(\frac{\partial^2\ell}{\partial\alpha\partial\theta_2}\right) & -E\left(\frac{\partial^2\ell}{\partial\alpha^2}\right) \end{pmatrix}\right|_{(\theta_1,\theta_2,0)}$$

The entries for $I(\theta_1, \theta_2, \alpha = 0)$ are $I_{11} = \frac{2n}{\theta_1^2} + \sum_{i=1}^{n}(y_{i1}+\theta_1+2)^{-2} - n(\theta_1+1)^{-2}(\bar{y}_1+3)$, $I_{12} = I_{21} = 0$, $I_{13} = I_{31} = \frac{\partial c_1}{\partial\theta_1}\sum_{i=1}^{n}(e^{-y_{i2}} - c_2)$, $I_{22} = \frac{2n}{\theta_2^2} + \sum_{i=1}^{n}(y_{i2}+\theta_2+2)^{-2} - n(\theta_2+1)^{-2}(\bar{y}_2+3)$, $I_{23} = I_{32} = \frac{\partial c_2}{\partial\theta_2}\sum_{i=1}^{n}(e^{-y_{i1}} - c_1)$ and $I_{33} = \sum_{i=1}^{n}\left[(e^{-y_{i1}} - c_1)(e^{-y_{i2}} - c_2)\right]^2$ where $c_t = \theta_t^2(1+\theta_t)^{-1}(\theta_t + 2 - e^{-1})(\theta_t - e^{-1} + 1)^{-2}$, $t = 1,2$ and

$$\frac{\partial c_t}{\partial\theta_t} = \frac{[3\theta_t^2 + 2\theta_t(2-e^{-1})][(1+\theta_t)(\theta_t - e^{-1}+1)] - 3\theta_t - e^{-1} + 3)[\theta_t^3 + (2-e^{-1})\theta_t^2]}{(1+\theta_t)^2(\theta_t - e^{-1}+1)^3}, t = 1,2.$$

When several models are available, one can compare the models' performance based on several likelihood measures. A regularly used measure is Akaike Information Criteria (AIC) which penalizes a model with larger number of parameters and is defined as $\text{AIC} = -2\ln L + 2p$, where $\ln L$ denotes the fitted log likelihood and p the number of parameters.

Application

Table 1 provides the flight aborts count data from 109 aircrafts, where random variables Y_1 and Y_2 respectively represent the number of flight aborts in the first and second consecutive six months of a one-year period (Mitchell and Paulson, 1981). Most observed frequencies provide (y_1, 0) and (0, y_2) data, indicating negative correlation between y_1 and y_2. Therefore, we fit BP (Lakshminarayana *et al*., 1999), BNB (Famoye, 2010) and BPL (Gomez-Deniz *et al*., 2012) distributions to the data since these distributions can be fitted to bivariate data with positive, zero or negative correlation.

The joint *p.m.f.* of *BP* (θ_1, θ_2, α) distribution is (Lakshminarayana *et al*., 1999):

$$Pr(Y_1 = y_1, Y_2 = y_2) = e^{-\theta_1-\theta_2}\frac{\theta_1^{y_1}\theta_2^{y_2}}{y_1!y_2!}$$
$$[1+\alpha(e^{-y_1} - e^{-d\theta_1})(e^{-y_2} - e^{-d\theta_2})]$$

$$y_1, y_2 = 0,1,2,..., \quad \theta_1, \theta_2 > 0$$

where, $d = 1 - e^{-1}$. The mean, variance and covariance are $E(Y_1) = Var(Y_1) = \theta_1$, $E(Y_2) = Var(Y_2) = \theta_2$ and $Cov(Y_1, Y_2) = \alpha\theta_1\theta_2 d^2 e^{-d(\theta_1+\theta_2)}$.

The joint p.m.f. of $BNB(\theta_1, \theta_2, a_1, a_2, \alpha)$ distribution is (Famoye, 2010):

$$P(Y_1 = y_1, Y_2 = y_2) = \binom{a_1^{-1} + y_1 - 1}{y_1}\theta_1^{y_1}$$
$$(1-\theta_1)^{a_1^{-1}}\binom{a_2^{-1} + y_2 - 1}{y_2}\theta_2^{y_2}(1-\theta_2)^{a_2^{-1}}$$
$$\times[1+\alpha(e^{-y_1} - c_1)(e^{-y_2} - c_2)]$$
$$y_1, y_2 = 0,1,2,..., \quad \theta_1, \theta_2 > 0, \quad 0 < a_1, a_2 < 1$$

Table 1. Observed and fitted values (flight aborts count data)

	Observed data	Fitted data		
(y_1, y_2)		BP	BNB	BPL
(0,0)	34	25.80	33.77	34.31
(0,1)	20	22.30	20.18	18.24
(0,2)	4	8.46	8.71	8.17
(0,3)	6	2.06	3.35	3.42
(0,4)	4	0.37	1.22	1.38
(1,0)	17	19.61	15.48	16.85
(1,1)	7	11.67	6.09	5.91
(1,2)	0	3.85	2.23	2.26
(1,3)	0	0.89	0.81	0.89
(1,4)	0	0.16	0.29	0.35
(2,0)	6	6.44	6.43	6.91
(2,1)	4	3.35	2.16	2.08
(2,2)	1	1.03	0.73	0.72
(2,3)	0	0.23	0.25	0.27
(2,4)	0	0.04	0.09	0.11
(3,0)	0	1.35	2.59	2.64
(3,1)	4	0.67	0.82	0.75
(3,2)	0	0.20	0.27	0.25
(3,3)	0	0.04	0.09	0.09
(3,4)	0	0.01	0.03	0.04
(4,0)	0	0.21	1.04	0.97
(4,1)	0	0.10	0.32	0.27
(4,2)	0	0.03	0.10	0.09
(4,3)	0	0.01	0.03	0.03
(4,4)	0	0.00	0.01	0.01
(5,0)	2	0.03	0.42	0.35
(5,1)	0	0.01	0.13	0.10
(5,2)	0	0.00	0.04	0.03
(5,3)	0	0.00	0.01	0.01
(5,4)	0	0.00	0.00	0.00

Table 2. Estimated parameters and AIC of BP, BNB and BPL distributions

Parameters	BP	BNB	BPL
θ_1	0.6129	0.4045	2.1165
θ_2	0.7131	0.3138	1.8607
a_1	-	1.0977	-
a_2	-	0.6305	-
α	-0.9290	-1.1109	-1.0363
Log *L*	-254.99	-244.27	-244.62
AIC	515.99	498.54	495.24

where, $c_t = E(e^{-Y_t}) = \left(\frac{1-\theta_t}{1-\theta_t e^{-1}}\right)^{a_t^{-1}}$, $t = 1,2$ and a_t, $t = 1,2$, is the dispersion parameter. The mean, variance and covariance are $E(Y_1) = a_1^{-1}\frac{\theta_1}{1-\theta_1}$, $Var(Y_1) = a_1^{-1}\frac{\theta_1}{(1-\theta_1)^2}$, $E(Y_2) = a_2^{-1}\frac{\theta_2}{1-\theta_2}$, $Var(Y_2) = a_2^{-1}\frac{\theta_2}{(1-\theta_2)^2}$ and $Cov(Y_1, Y_2) = \alpha c_1 c_2 A_1 A_2$ where $A_t = \frac{a_t^{-1}\theta_t e^{-1}}{1-\theta_t e^{-1}} - \frac{a_t^{-1}\theta_t}{1-\theta_t}$, $t = 1,2$. Therefore, the correlation coefficient for *BP*, *BNB* and *BPL* distributions can be positive, zero or negative, depending on the value of multiplicative factor parameter, α. For comparison purpose, Table 1 also provides the fitted values from *BP*, *BNB* and *BPL* distributions.

Table 2 provides the estimated parameters, log likelihood and *AIC* for *BP*, *BNB* and *BPL* distributions. It can be seen that all distributions provide negative value for α, indicating negative correlation. Even though both BNB and BPL distributions produce similar log likelihood, the number of parameters for BPL distribution is less and thus, producing smaller *AIC*. Based on *AIC*, *BPL* distribution provides the best fit for the data.

Conclusion

In this study, *BPL* distribution has been fitted to a sample of bivariate count data. Based on the results, *BPL* distribution provides better fit than *BP* and *BNB* distributions, indicating that the distribution can be used as an alternative for fitting dependent and over-dispersed count data, with either positive or negative correlation.

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Author's Contributions

Hossein Zamani: Suggested BPL distribution and developed R program for fitting BPL distribution.

Pouya Faroughi: Developed R program for fitting BP, BNB and BPL distributions, coordinated the data analysis and contributed to the writing of the manuscript.

Noriszura Ismail: Contributed to the writing of the manuscript, designed the research plan and carried out the overall editing of manuscript.

Ethics

This article is original and contains unpublished material. The corresponding author confirms that all of the other authors have read and approved the manuscript and no ethical issues involved.

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