

ON THE PRODUCT AND RATIO OF TWO GENERALIZED ORDER STATISTICS FROM THE GENERALIZED BURR TYPE-II DISTRIBUTION

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ABSTRACT

In this study, the probability density functions of the product and quotient of two generalized order statistics from the generalized Burr Type-XII distribution have been derived, using the Mellin transform and its inverse. Several interesting special cases have also been obtained such as the product and quotient of extreme generalized order statistics and the product and quotient of consecutive generalized order statistics.

Keywords: Generalized Order Statistics, Mellin Transform Technique, Hyper Geometric Function, Fox H-Function, Generalized Burr Type-XII Distribution

1. INTRODUCTION

The Exponentiated distributions can be obtained by two common techniques: Firstly, Powering a positive real number to the Cumulative Distribution Function (CDF), i.e., if we have CDF $F(x)$ of any random variable X , then the function $F(z) = [F(z)]^\theta$, $\theta > 0$, is called an exponentiated distribution Gupta and Kundu (1999; 2001); Mudholkar and Srivastava (1993) and Nadarajah (2005). Secondly, using the formula $F(z) = 1 - [1 - F(z)]^\theta$ Nadarajah (2005).

The aim of this study is to study a distribution which generalizes the standard Burr Type-XII distribution using the second technique. Thus, the Probability Density Function (PDF) and the CDF of the exponentiated Burr Type XII distribution (EB), are given respectively as Equation 1 and 2:

$$f(x) = \frac{\alpha\beta\theta x^{-\alpha-1}(1+x^{-\alpha})^{-\beta-1}}{[1-(1+x^{-\alpha})^{-\beta}]^{\theta-1}} \quad (1)$$

$$F(x) = 1 - [1 - (1 + x^{-\alpha})^{-\beta}]^\theta \quad (2)$$

$\alpha, \beta > 0$ are shape parameters and θ is called the exponentiated parameter.

The Burr type-XII model has been considered by Burr (1942), as a new lifetime distribution and has been

studied extensively in the literature as an appropriate and useful failure model to the applied statistics. Evans and Simon (1975) discussed the Burr distribution as a failure model and derived the Maximum Likelihood Estimators (MLE) for its parameters. Evans and Ragab (1983) discussed the Bayes estimators for Burr distribution parameters based on type-II censored samples. Wu and Yu (2005) proposed pivotal quantities to test the shape parameter and established its confidence interval based on censored data. Soliman (2002) derived the MLE and Bayes estimates for the parameters. Xiuchun *et al.* (2007) derived the empirical Bayes estimator of the reliability based on progressively type-II censored samples and Wang and Shi (2010) derived the empirical Bayes based on record values. For intensive review for the Burr type-XII distribution, Rodriguez (1977).

However, in recent years, this distribution has been used in a variety of fields such as business Rodriguez and Taniguchi (1980). Economics and finance McDonald and Richards (1987). Hydrology, Mielke and Johanson (1974). Quality assurance, Burr (1976). Medicine, Wingo (1993) and Mineralogy, Cook and Johnson (1986). Lee *et al.* (2009) obtained the Bayes and empirical Bayes estimators for the reliability. Thus several aspects of the distribution have been covered by researchers, but a comprehensive statistical analysis on the distribution is still lacking.

The distributions of the product and the quotient of two random variables are widely used in many areas of statistical analysis. An important example of

products of random variables is the case of investment in a number of different overseas markets. An important example for the quotient of order statistics is the stress-length model within the context of reliability. It describes the life of a component which has a random stress Y and is subject to random stress X . The component fails at the instant that stress applied to it exceeds the strength. The distribution of product and quotient have been studied by several authors, Subrahmaniam (1970); Wallgren (1980); Malik and Trudel (1982); Tang and Gupta (1984); Nadarajah (2005); Nadarajah and Gupta (2005a; 2005b); Nadarajah and Kotz (2007); Aleem (2008); Garg (2009) and Nagar and Valencia (2011). They are met in the problem of selection and ranking rules, which is described by Gulati (1970) and Malik (1970). They also can occur when the random variables have dimensions of a ratio such as cost of a structure per pound of payload or fuel consumption per mile. Subrahmaniam (1970) has made the study of product and quotient of order statistics from uniform distribution and exponential distribution. Malik and Trudel (1982) studied the cases when the order statistics are from Pareto, power and Weibull distributions. Aleem (2008) studied the quotient of two generalized order statistics from the Weibull distribution. Garg (2009) has studied the product and quotient of two generalized order statistics from Kumaraswami distribution. In this study, the distributions of the product and quotient of two generalized order statistics from the exponentiated Burr Type-XII distribution have been derived.

The concept of the Generalized Order Statistics (GOS) is introduced by Kamps (1995) as a unified approach to ordinary OS, record values and k -th record values, which can be outlined as:

The random variables $X(1, n, m, k), \dots, X(n, n, m, k)$ be GOS from an absolutely continuous CDF $F(x)$ and PDF $f(x)$, with noting that $(X(0, n, m, k) = 0, \geq 1, m \in \mathbb{R}^{n-1})$, then their joint PDF can be written in the form Equation 3:

$$f(x_1, x_2, \dots, x_n) = k \prod_{i=1}^{n-1} \gamma_i f(x_i) \cdot [1 - F(x_i)]^m [1 - F(x_n)]^{k-1} f(x_n) \quad (3)$$

On the cone $F^{-1}(0) < x_1 < \dots < x_n < F^{-1}(1)$ of \mathbb{R}^n , where $\gamma_i = k + (n - i)(m + 1)$ and $\gamma_n = k > 0$.

If $m = 0$ and $k = 1$, then (1.3) is the joint PDF of the ordinary OS.

The joint PDF of $X(i, n, m, k)$ and $X(j, n, m, k)$, $i < j$ is given by Equation 4:

$$f_{i,j}(x, y) = \frac{c_{j-1}}{(i-1)!(j-i-1)!} (1 - F(x))^m f(x) g_m^{i-1}(F(x)) \cdot [h_m(F(y)) - h_m(F(x))]^{j-i-1} [1 - F(y)]^{j-1} f(y), \quad (4)$$

For $x < y$.

Here $C_{j-i} = \prod_{i=1}^j \gamma_i$ $j = 1, 2, \dots, n$ and $x \in [0, 1]$, $g_m(x) =$

$$h_m(x) - h_m(0), \quad h_m(x) = \begin{cases} -\frac{1}{m+1} (1-x)^{m+1}, & m \neq -1 \\ -\log(1-x), & m = -1 \end{cases}$$

1.1. The Mellin Transform

Epstein (1948) was first to suggest a systematic approach to the study of products and quotients of independent random variables by using a Mellin transform technique.

The Mellin transform of $f(x, t)$ is defined as Equation 5:

$$M_{s_1, s_2}(f(x, y)) = \int_0^y \int_0^x x^{s_1-1} y^{s_2-1} f(x, y) dx dy \quad (5)$$

where, S_1 and S_2 are complex variables. Under suitable conditions discussed by Fox (1957), it possesses the inverse as Equation 6:

$$f(x, y) = \frac{1}{(2\pi i)^2} \int_{h-i\infty}^{h+i\infty} \int_{k-i\infty}^{k+i\infty} x^{-s_1} y^{-s_2} M_{s_1, s_2}(f(x, y)) ds_1 ds_2 \quad (6)$$

With the paths of integration being two lines parallel to the imaginary axis and to the right from the origin in the Argand plane.

We will study two cases:

- When $S_1 = S_2 = t$, we get:

$M_{t,t}(g(u)) = M(t|u)$, which corresponds to the Mellin transform for the PDF of the product $U = XY$. Thus Equation 7:

$$g(u) = \frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} u^{-t} M(t|u) dt \quad (7)$$

- When $S_1 = t$ and $S_2 = -t + 2$, we get $M_{t, -t+2}(h(z)) = M(t|z)$, which corresponds to the Mellin transform for the PDF of the quotient $Z = X/Y$. Thus Equation 8:

$$h(z) = \frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} z^{-t} M(t|z) dt \quad (8)$$

1.2. The Fox H-Function

The H-function is a generalization of the G-function and also it is defined, in the literature, as a Mellin-Barnes integral, Gradshteyn *et al.* (2007). The definitions of these functions will be defined as follows:

The G-function is defined as Equation 9:

$$G_{p,q}^{m,n} \left[z \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right] = \frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} z^{-t} g(t) dt \tag{9}$$

$$g(t) = \frac{\prod_{j=1}^m \Gamma(b_j + t) \prod_{j=1}^n \Gamma(1 - a_j - t)}{\prod_{j=m+1}^q \Gamma(1 - b_j - t) \prod_{j=n+1}^p \Gamma(a_j + t)}$$

The H-function is defined as Equation 10:

$$H_{p,q}^{m,n} \left[z \begin{matrix} (a_1, \alpha_1), \dots, (a_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{matrix} \right] = \frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} z^t h(t) dt \tag{10}$$

$$h(t) = \frac{\prod_{j=1}^m \Gamma(b_j + \beta_j t) \prod_{j=1}^n \Gamma(1 - a_j - \alpha_j t)}{\prod_{j=m+1}^q \Gamma(1 - b_j - \beta_j t) \prod_{j=n+1}^p \Gamma(a_j + \alpha_j t)}$$

m, n, p and q are non-negative integers satisfying, $0 \leq n \leq p$, $1 \leq m \leq q$; $\alpha_j (j = 1, \dots, p)$ and $\beta_j (j = 1, \dots, q)$ are assumed to be positive quantities for standardization Purpose.

The inverse relation for the Fox function will be given as Equation 11:

$$H_{p,q}^{m,n} \left[z \begin{matrix} (a_i, \alpha_i), i = 1, p \\ (b_j, \beta_j), j = 1, q \end{matrix} \right] = H_{p,q}^{m,n} \left[\frac{1}{z} \begin{matrix} (1 - b_j, \beta_j), j = 1, q \\ (1 - a_i, \alpha_i), i = 1, p \end{matrix} \right] \tag{11}$$

1.3. The Distribution of Generalized Order Statistics

1.3.1. The Distribution of Product of Two Generalized Order Statistics

Theorem 1

Let $X(i, n, m, k)$ and $X(j, n, m, k)$ be the *i*th and *j*th generalized order statistics with ($i < j$), based on a

random sample of size *n* from the generalized exponentiated Burr Type-XII distribution, then the PDF of the product $U = X(i, n, m, k).X(j, n, m, k)$ is given by Equation 12:

$$g(u) = \frac{(\theta\beta)^2 c_{j-1}}{(i-1)!(j-i-1)!(m+1)^{j-2}} \sum_{r,s}^* \sum_{l_1, l_2}^* \frac{1}{\Gamma(\beta(l_2 + 1) + 1)} H_{3,3}^{2,2} \left[\frac{1}{u} \begin{matrix} \left(1 + \frac{1}{\alpha}, \quad 2 + \frac{1}{\alpha} + \beta, \quad 1 + \frac{2}{\alpha} \right) \\ -\beta(l_1 + 1), \frac{1}{\alpha}, \quad (l_1 + 1), \frac{1}{\alpha}, \quad -\beta(l_1 + l_2 + 1), \frac{2}{\alpha} \\ \left(1 + \frac{1}{\alpha}, \frac{1}{\alpha}, \quad 2 + \frac{2}{\alpha}, \frac{2}{\alpha}, \quad \left(\frac{1}{\alpha} - \beta(l_1 + 1), \frac{1}{\alpha} \right) \right) \end{matrix} \right] \tag{12}$$

where, $0 < u \leq \infty$, $H[.]$ is the Fox H- function defined by (10), $1 \leq i < j \leq n$ and the symbol c_{j-1} as defined in section (1).

Proof

For simplicity, let *X* and *Y* be the *i*th and *j*th GOS with ($i < j$) from the exponentiated Burr Type-XII distribution (1). Thus, substituting (1) and (2) in (4), the joint PDF of these generalized order statistics is given by Equation 13:

$$f(x, y) = \frac{c_{j-1} (\alpha\beta\theta)^2}{(i-1)!(j-i-1)!(m+1)^{j-2}} \sum_{r=0}^{i-1} \sum_{s=0}^{j-i-1} (-1)^{r+s} \binom{i-1}{r} \binom{j-i-1}{s} \sum_{l_1=0}^{\theta(m+1)(j+r-i-s)-1} \sum_{l_2=0}^{\theta((m+1)s+\gamma_j)-1} (-1)^{l_1+l_2} \binom{\theta(m+1)(j+r-i-s)-1}{l_1} \binom{\theta((m+1)s+\gamma_j)-1}{l_2} x^{-\alpha-1} y^{-\alpha-1} (1+x^{-\alpha})^{-\beta(l_1+1)-1} (1+y^{-\alpha})^{-\beta(l_2+1)-1} \tag{13}$$

The Mellin transform of (13) is given by:

$$M_{s_1, s_2} (f(x, y)) = K(\alpha\beta\theta)^2 \sum_{r,s}^* \sum_{l_1, l_2}^* \int_0^\infty y^{s_2 - \alpha - 2} (1 + y^{-\alpha})^{-\beta(l_2 + 1) - 1} I(y) dy$$

Where:

$$K = \frac{c_{j-1}}{(i-1)!(j-i-1)!(m+1)^{j-2}},$$

$$\sum_{r,s}^* = \sum_{r=0}^{i-1} \sum_{s=0}^{j-i-1} (-1)^{r+s} \binom{i-1}{r} \binom{j-i-1}{s}^{(j-i-1)},$$

$$\sum_{l_1, l_2}^* = \sum_{l_1=0}^{\theta(m+1)(j+r-i-s)-1} \sum_{l_2=0}^{\theta(m+1)s+\gamma_j-1} (-1)^{l_1+l_2} \binom{\theta(m+1)(j+r-i-s)-1}{l_1} \binom{\theta(m+1)s+\gamma_j-1}{l_2}$$

And:

$$I(y) = \int_0^y x^{s_1-\alpha-2} (1+x^{-\alpha})^{-\beta(l_1+1)-1} dx.$$

Let $v = x^{-\alpha}$ and $w = y^{-\alpha}$, thus:

$$I(y) = \frac{1}{\alpha} \int_w^{\frac{y}{w}} t^{-\frac{s_1-1}{\alpha}} (1+t)^{-\beta(l_1+1)-1} dt$$

$$= \frac{1}{\alpha} \frac{w^{-\frac{s_1-1}{\alpha}-\beta(l_1+1)}}{(\frac{s_1-1}{\alpha} + \beta(l_1+1))} {}_2F_1(\beta(l_1+1)+1, \frac{s_1-1}{\alpha} + \beta(l_1+1); \frac{s_1-1}{\alpha} + \beta(l_1+1)+1; -\frac{1}{w})$$

Gradshteyn *et al.* (2007), thus Equation 14:

$$M_{s_1, s_2}(f(x, y)) = K(\theta\beta)^2 \sum_{r,s}^* \sum_{l_1, l_2}^* \frac{1}{\alpha} \frac{w^{-\frac{s_1-1}{\alpha}-\beta(l_1+1)}}{(\frac{s_1-1}{\alpha} + \beta(l_1+1))} \int_0^w (1+w)^{-\beta(l_2+1)-1} (1+w)^{-\beta(l_1+1)-1} dw$$

$${}_2F_1(\beta(l_1+1)+1, \frac{s_1-1}{\alpha} + \beta(l_1+1); \frac{s_1-1}{\alpha} + \beta(l_1+1)+1; -\frac{1}{w})$$

Let $v = -\frac{1}{w}$ and using the integral in Gradshteyn *et al.* (2007), the Mellin transform (14) can be written as Equation 15:

$$M_{s_1, s_2}(f(x, y)) = K(\theta\beta)^2 \sum_{r,s}^* \sum_{l_1, l_2}^* \frac{1}{\alpha} \frac{\Gamma(\frac{s_1+s_2-2}{\alpha} + \beta(l_1+l_2+2)) \Gamma(-\frac{s_1+s_2-2}{\alpha} + 2) \Gamma(-\frac{s_2-1}{\alpha} + 1)}{\Gamma(\beta(l_2+1)+1) \Gamma(\beta(l_1+1) - \frac{s_2-1}{\alpha} + 2)}$$

By setting $s_1 = t$ and $s_2 = t$ in (15), we get $M_{t,t}(g(u)) = M(t|u)$, which corresponds to the Mellin transform for the PDF of the product $U = XY$. Thus Equation 16:

$$M_{t,t}(g(u)) = M(t|u) = K(\theta\beta)^2 \sum_{r,s}^* \sum_{l_1, l_2}^* \frac{1}{\alpha} \frac{\Gamma(\frac{2t-2}{\alpha} + \beta(l_1+l_2+2)) \Gamma(-\frac{2t-2}{\alpha} + 2) \Gamma(-\frac{t-1}{\alpha} + 1)}{\Gamma(\beta(l_2+1)+1) \Gamma(\beta(l_1+1) - \frac{t-1}{\alpha} + 2)}$$

$$= K(\theta\beta)^2 \sum_{r,s}^* \sum_{l_1, l_2}^* \frac{\Gamma(\beta(l_1+1) + \frac{t-1}{\alpha}) \Gamma(\frac{2t-2}{\alpha} + \beta(l_1+l_2+2)) \Gamma(-\frac{2t-2}{\alpha} + 2) \Gamma(-\frac{t-1}{\alpha} + 1)}{\Gamma(\beta(l_1+1) + \frac{t-1}{\alpha} + 1) \Gamma(\beta(l_2+1)+1) \Gamma(\beta(l_1+1) - \frac{t-1}{\alpha} + 2)}$$

Thus, using (7) with (16), we get the PDF of the product U as:

$$g(u) = K(\theta\beta)^2 \sum_{r,s}^* \sum_{l_1, l_2}^* \frac{1}{\Gamma(\beta(l_2+1)+1)} \frac{\Gamma(\beta(l_1+1) + \frac{t-1}{\alpha}) \Gamma(\frac{2t-2}{\alpha} + \beta(l_1+l_2+2)) \Gamma(-\frac{2t-2}{\alpha} + 2) \Gamma(-\frac{t-1}{\alpha} + 1)}{2\pi i \int_{h-i\infty}^{h+i\infty} u^{-t} \Gamma(\beta(l_1+1) + \frac{t-1}{\alpha} + 1) \Gamma(\beta(l_1+1) - \frac{t-1}{\alpha} + 2) dt}$$

$$g(u) = K(\theta\beta)^2 \sum_{r,s}^* \sum_{l_1, l_2}^* \frac{1}{\Gamma(\beta(l_2+1)+1)} \left[\begin{matrix} (-\frac{1}{\alpha}, \frac{1}{\alpha}), (-1 - \frac{2}{\alpha}, \frac{2}{\alpha}), (1 - \frac{1}{\alpha} + \beta(l_1+1), \frac{1}{\alpha}) \\ (\beta(l_1+1) - \frac{1}{\alpha}, \frac{1}{\alpha}), \\ (-1 - \frac{1}{\alpha} - \beta(l_1+1), \frac{1}{\alpha}), (-\frac{2}{\alpha} + \beta(l_1+l_2+2), \frac{2}{\alpha}) \end{matrix} \right]$$

Using the inverse relation (11) we get the PDF of the product (12).

1.4. The Distribution of Quotient of Two Generalized Order Statistics

Theorem 2:

Let $X(i, n, m, k)$ and $X(j, n, m, k)$ be the i th and j th generalized order statistics with $(i < j)$, based on a

random sample of size n from the generalized Burr Type-XII distribution, then the PDF of the quotient $Z = X(i, n, m, k)/X(j, n, m, k)$ is given by Equation 17:

$$h(z) = \frac{(\theta\beta)^2 c_{j-1}}{(i-1)!(j-i-1)!(m+1)^{j-2}} \sum_{r,s}^* \sum_{l_1, l_2}^* \frac{\Gamma(\beta(l_1 + l_2 + 2))}{\Gamma(\beta(l_2 + 1) + 1)} \cdot H_{2,2}^{2,0} \left[\frac{1}{z} \left| \begin{matrix} (\frac{1}{\alpha}, \frac{1}{\alpha}), & (1 + \frac{1}{\alpha} - \beta(l_1 + 1), \frac{1}{\alpha}) \\ (\frac{1}{\alpha} - \beta(l_1 + 1), \frac{1}{\alpha}), & (-1 + \frac{1}{\alpha} - \beta(l_1 + 1), \frac{1}{\alpha}) \end{matrix} \right. \right] \quad (17)$$

where, $0 < Z \leq \infty$, $1 \leq i < j \leq n$ and the symbol c_{j-1} as defined in section 1.

Proof

Using the Mellin transform (15) by setting $s_1 = t$ and $s_2 = t + 2$, we get $M_{t, t+2}(h(z)) = M(t|z)$, which corresponds to the Mellin transform for the PDF of the quotient $Z = X/Y$. Thus Equation 18:

$$M_{t, t+2}(h(z)) = M(t|z) = K(\theta\beta)^2 \sum_{r,s}^* \sum_{l_1, l_2}^* \frac{1}{\frac{t-1}{\alpha} + \beta(l_1 + 1)} \frac{\Gamma(\beta(l_1 + l_2 + 2))\Gamma(2)\Gamma(\frac{t-1}{\alpha} + 1)}{\Gamma(\beta(l_2 + 1) + 1)\Gamma(\beta(l_1 + 1) + \frac{t-1}{\alpha} + 2)} \quad (18)$$

Thus, from (8) with (19) we get the PDF of the quotient as:

$$h(z) = K(\alpha\theta)^2 \sum_{r,s}^* \sum_{l_1, l_2}^* \frac{\Gamma(\beta(l_1 + l_2 + 2))}{\Gamma(\beta(l_2 + 1) + 1)} \frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} z^{-t} \frac{\Gamma(\frac{t-1}{\alpha} + 1)\Gamma(\beta(l_1 + 1) + \frac{t-1}{\alpha})}{\Gamma(\beta(l_1 + 1) + \frac{t-1}{\alpha} + 1)\Gamma(\beta(l_1 + 1) + \frac{t-1}{\alpha} + 2)} dt$$

$$h(z) = K(\theta\beta)^2 \sum_{r,s}^* \sum_{l_1, l_2}^* \frac{\Gamma(\beta(l_1 + l_2 + 2))}{\Gamma(\beta(l_2 + 1) + 1)} \cdot H_{2,2}^{2,0} \left[z \left| \begin{matrix} (1 - \frac{1}{\alpha} + \beta(l_1 + 1), \frac{1}{\alpha}), & (2 - \frac{1}{\alpha} + \beta(l_1 + 1), \frac{1}{\alpha}) \\ (1 - \frac{1}{\alpha}, \frac{1}{\alpha}), & (\beta(l_1 + 1) - \frac{1}{\alpha}, \frac{1}{\alpha}) \end{matrix} \right. \right]$$

Using the inverse relation (11) we get the PDF of the quotient (18).

Corollary 1

Let $I = 1$ and $j = n$ in (12) and (18) we get the density functions of the product and quotient of the extreme generalized order statistics from the exponentiated Burr Type XII distribution respectively as:

$$g(u) = \frac{(\theta\beta)^2 c_{n-1}}{(n-2)!(m+1)^{n-2}} \sum_{r,s}^{**} \sum_{l_1, l_2}^{**} \frac{1}{\Gamma(\beta(l_2 + 1) + 1)} \cdot H_{3,3}^{2,2} \left[\frac{1}{u} \left| \begin{matrix} (1 + \frac{1}{\alpha} - \beta(l_1 + 1), \frac{1}{\alpha}), & (2 + \frac{1}{\alpha} + \beta(l_1 + 1), \frac{1}{\alpha}), \\ (1 + \frac{2}{\alpha} - \beta(l_1 + l_2 + 1), \frac{2}{\alpha}), \\ (1 + \frac{1}{\alpha}, \frac{1}{\alpha}), & (2 + \frac{2}{\alpha}, \frac{2}{\alpha}), & (\frac{1}{\alpha} - \beta(l_1 + 1), \frac{1}{\alpha}) \end{matrix} \right. \right]$$

$$h(z) = \frac{(\theta\beta)^2 c_{n-1}}{(n-2)!(m+1)^{n-2}} \sum_{r,s}^{**} \sum_{l_1, l_2}^{**} \frac{\Gamma(\beta(l_1 + l_2 + 2))}{\Gamma(\beta(l_2 + 1) + 1)} \cdot H_{2,2}^{2,0} \left[\frac{1}{z} \left| \begin{matrix} (\frac{1}{\alpha}, \frac{1}{\alpha}), & (1 + \frac{1}{\alpha} - \beta(l_1 + 1), \frac{1}{\alpha}) \\ (\frac{1}{\alpha} - \beta(l_1 + 1), \frac{1}{\alpha}), & (-1 + \frac{1}{\alpha} - \beta(l_1 + 1), \frac{1}{\alpha}) \end{matrix} \right. \right]$$

Where:

$$\sum_{s=0}^{n-2} (-1)^s \binom{n-2}{s}$$

$$\sum_{l_1, l_2}^{**} = \sum_{l_1=0}^{\theta(m+1)(n-s-1)-1} \sum_{l_2=0}^{\theta(m+1)s+k-1} (-1)^{l_1+l_2} \binom{\theta(m+1)(n-s-1)-1}{l_1} \binom{\theta(m+1)s+k-1}{l_2}$$

Setting $\theta = 1$, we get the density functions of the product and quotient of the extreme generalized order statistics from the standard Burr Type XII distribution.

Corollary 2

Let $j = i + 1$ in (4.1) and (5.1) we get the density functions of the product and quotient of the consecutive generalized order statistics ($i < j$) from the exponentiated Burr Type XII distribution respectively as:

$$g(u) = \frac{(\theta\beta)^2 c_i}{(i-1)!(m+1)^{i-1}} \sum_r^{**} \sum_{l_1, l_2}^{***} \frac{1}{\Gamma(\beta(l_2 + 1) + 1)} \cdot H_{3,3}^{2,2} \left[\frac{1}{u} \left| \begin{matrix} (1 + \frac{1}{\alpha} - \beta(l_1 + 1), \frac{1}{\alpha}), & (2 + \frac{1}{\alpha} + \beta(l_1 + 1), \frac{1}{\alpha}), \\ (1 + \frac{2}{\alpha} - \beta(l_1 + l_2 + 1), \frac{2}{\alpha}), \\ (1 + \frac{1}{\alpha}, \frac{1}{\alpha}), & (2 + \frac{2}{\alpha}, \frac{2}{\alpha}), & (\frac{1}{\alpha} - \beta(l_1 + 1), \frac{1}{\alpha}) \end{matrix} \right. \right]$$

$$h(z) = \frac{(\theta\beta)^2 c_i}{(i-1)!(m+1)^{i-1}} \sum_r^{**} \sum_{l_1, l_2}^{***} \frac{\Gamma(\beta(l_1 + l_2 + 2))}{\Gamma(\beta(l_2 + 1) + 1)}$$

$$\cdot H_{2,2}^{2,0} \left[\frac{1}{z} \left| \begin{matrix} (\frac{1}{\alpha}, \frac{1}{\alpha}), & (1 + \frac{1}{\alpha} - \beta(l_1 + 1), \frac{1}{\alpha}) \\ (\frac{1}{\alpha} - \beta(l_1 + 1), \frac{1}{\alpha}), & (-1 + \frac{1}{\alpha} - \beta(l_1 + 1), \frac{1}{\alpha}) \end{matrix} \right. \right]$$

Where:

$$\sum_r^{**} = \sum_{r=0}^{i-1} (-1)^r \binom{i-1}{r}$$

$$\sum_{l_1, l_2}^{***} = \sum_{l_1=0}^{\theta(m+1)(r+1)-1} \sum_{l_2=0}^{\theta\gamma_{i+1}-1} (-1)^{l_1+l_2} \binom{\theta(m+1)(r+1)-1}{l_1} \binom{\theta\gamma_{i+1}-1}{l_2}$$

Setting $\theta = 1$, we get the density functions of the product and quotient for consecutive generalized order statistics from the standard Burr Type XII distribution.

Corollary 3

When $\theta = 1$, (12) and (18) reduce to the density functions of the product and quotient of the i th and j th generalized order statistics from the Burr Type XII distribution respectively as:

$$g(u) = \frac{\beta^2 c_{j-1}}{(i-1)!(j-i-1)!(m+1)^{j-2}} \sum_{r,s}^* \sum_{l_1, l_2}^/ \frac{1}{\Gamma(\beta(l_2 + 1) + 1)}$$

$$H_{3,3}^{2,2} \left[\frac{1}{u} \left| \begin{matrix} (1 + \frac{1}{\alpha} - \beta(l_1 + 1), \frac{1}{\alpha}), & (2 + \frac{1}{\alpha} + \beta(l_1 + 1), \frac{1}{\alpha}), \\ (1 + \frac{2}{\alpha} - \beta(l_1 + l_2 + 1), \frac{2}{\alpha}) \\ (1 + \frac{1}{\alpha}, \frac{1}{\alpha}), & (2 + \frac{2}{\alpha}, \frac{2}{\alpha}), & (\frac{1}{\alpha} - \beta(l_1 + 1), \frac{1}{\alpha}) \end{matrix} \right. \right]$$

$$h(z) = \frac{\beta^2 c_{j-1}}{(i-1)!(j-i-1)!(m+1)^{j-2}} \sum_{r,s}^* \sum_{l_1, l_2}^/ \frac{\Gamma(\beta(l_1 + l_2 + 2))}{\Gamma(\beta(l_2 + 1) + 1)}$$

$$H_{2,2}^{2,0} \left[\frac{1}{z} \left| \begin{matrix} (\frac{1}{\alpha}, \frac{1}{\alpha}), & (1 + \frac{1}{\alpha} - \beta(l_1 + 1), \frac{1}{\alpha}) \\ (\frac{1}{\alpha} - \beta(l_1 + 1), \frac{1}{\alpha}), & (-1 + \frac{1}{\alpha} - \beta(l_1 + 1), \frac{1}{\alpha}) \end{matrix} \right. \right]$$

Where:

$$\sum_{l_1, l_2}^/ = \sum_{l_1=0}^{(m+1)(r+1)-1} \sum_{l_2=0}^{\gamma_{i+1}-1} (-1)^{l_1+l_2} \binom{(m+1)(r+1)-1}{l_1} \binom{\gamma_{i+1}-1}{l_2}$$

Corollary 4

When $\theta = 1$ and $m = 0$ and $k = 1$ (12) and (18) reduce to the density functions of the product and the quotient of the i th and j th order statistics from the ordinary Burr Type XII distribution respectively as:

$$g(u) = \frac{\beta^2 n!}{(i-1)!(j-i-1)!(n-j)!} \sum_{r,s}^* \sum_{l_1, l_2}^/ \frac{1}{\Gamma(\beta(l_2 + 1) + 1)}$$

$$H_{3,3}^{2,2} \left[\frac{1}{u} \left| \begin{matrix} (1 + \frac{1}{\alpha} - \beta(l_1 + 1), \frac{1}{\alpha}), & (2 + \frac{1}{\alpha} + \beta(l_1 + 1), \frac{1}{\alpha}), \\ (1 + \frac{2}{\alpha} - \beta(l_1 + l_2 + 1), \frac{2}{\alpha}) \\ (1 + \frac{1}{\alpha}, \frac{1}{\alpha}), & (2 + \frac{2}{\alpha}, \frac{2}{\alpha}), & (\frac{1}{\alpha} - \beta(l_1 + 1), \frac{1}{\alpha}) \end{matrix} \right. \right]$$

$$h(z) = \frac{\beta^2 n!}{(i-1)!(j-i-1)!(n-j)!} \sum_{r,s}^* \sum_{l_1, l_2}^/ \frac{\Gamma(\beta(l_1 + l_2 + 2))}{\Gamma(\beta(l_2 + 1) + 1)}$$

$$\cdot H_{2,2}^{2,0} \left[\frac{1}{z} \left| \begin{matrix} (\frac{1}{\alpha}, \frac{1}{\alpha}), & (1 + \frac{1}{\alpha} - \beta(l_1 + 1), \frac{1}{\alpha}) \\ (\frac{1}{\alpha} - \beta(l_1 + 1), \frac{1}{\alpha}), & (-1 + \frac{1}{\alpha} - \beta(l_1 + 1), \frac{1}{\alpha}) \end{matrix} \right. \right]$$

Where:

$$\sum_{l_1, l_2}^/ = \sum_{l_1=0}^{(j+r-i-s-1)n-j+s} \sum_{l_2=0}^{(j+r-i-s-1)(n-j+s)} (-1)^{l_1+l_2} \binom{(j+r-i-s-1)n-j+s}{l_1} \binom{(j+r-i-s-1)(n-j+s)}{l_2}$$

Garg (2009) when $\alpha = -a$ and $\beta = -b$.

2. CONCLUSION

In this study, we have proposed a new family of distributions called exponentiated Burr type-XII distribution. We get the probability density functions for the product and Quotient of two generalized statistics from this distribution. We have derived some interesting special cases such as the product and quotient of extreme generalized order statistics and the product and quotient of consecutive generalized order statistics.

These functions are useful in the investment in a number of different overseas markets and the problem of selection and ranking rules.

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