Journal of Mathematics and Statistics 6 (4): 409-415, 2010
ISSN 1549-3644

Smooth Neighborhood Structures in a Smooth Topological Spaces

A.A. Ramadan and M.A. Abdel-Sattar
Department of Mathematics, Faculty of Science,
Beni-Suef University, Beni-Suef, Egypt

Abstract: Problem Statement: Various concepts related to a smooth topological spaces have been introduced and relations among them studied by several authors (Chattopadhyay, Ramadan, etc). **Conclusion/Recommendations**: In this study, we presented the notions of three sorts of neighborhood structures of a smooth topological spaces and give some of their properties which are results by Ying extended to smooth topological spaces.

Key words: Fuzzy smooth topology, smooth neighborhood structures

INTRODUCTION

Šostak (1985) introduced the fuzzy topology as an extension of Chang (1968) fuzzy topology. It has been developed in many directions (Ramadan, 1992; Chattopadhyay and Samanta, 1993; EL Gayyar *et al*., 1994; Höhle and Rodabaugh, 1998; Kubiak and Šostak, 1997; Demirici, 1997; Ramadan *et al*., 2001; 2009; Abdel-Sattar, 2006).

Ying (1994) studied the theory of neighborhood systems in fuzzy topology with the method used to develop fuzzifying topology (Ying, 1991) by treating the membership relation as a fuzzy relation. In this study, we generate the structures of neighborhood systems in a smooth topology with the method used in (Ying, 1991), by using fuzzy sets and fuzzy points.

Notions and preliminaries: The class of all fuzzy sets on a universal set X will be denote by L^X, where L is the special lattice and L = ([0,1], ≤). Also, $L_0 = (0,1]$ and $L_1 = [0, 1)$.

Definition 1: Pu and Liu (1980) a fuzzy set in X is called a fuzzy point iff it takes the value 0 for all y ∈ X, except one, say x ∈ X. If its value at x is λ (0 < λ≤1) we denote this fuzzy point by x_λ, where the point x is called its support. The fuzzy point is said to be contained in a fuzzy set A, or belong to A, denoted by $x_\lambda \in A$, iff λ≤A(x). Evidently, every fuzzy set A can be expressed as the union of all fuzzy points which belong to A.

Definition 2: Ying (1991) Let X be a non-empty set. Let x_λ be a fuzzy point in X and let A be a fuzzy subset of X. Then the degree to which x_λ belongs to A is:

$$m(x_\lambda, \cup_{i\in\Gamma} A_i) = \vee_{i\in\Gamma} m(x_\lambda, A_i)$$

Obviously, we have the following properties:

(1) m(x, A) = A(x)

(2) $m(x_\lambda, A) = 1$ iff $x_\lambda \in A$, $m(x_\lambda, A) = 0$ iff λ = 1 and A(x) = 0

(3) $m(x_\lambda, \text{È}_{i\text{Î}\Gamma} A_i) = \text{Ú}_{i\text{Î}\Gamma} m(x_\lambda, A_i)$, (generalized multiple choice principles)

Definition 3: Ying (1991) let (X,τ) be a fuzzy topological space (fts, for short), let e be a fuzzy point in X and let A be a fuzzy subset of X. Then the degree to which A is a neighborhood of e is defined by:

$$N_e(A) = \sup\{m(e,B) : B \in \tau, B \subseteq A\}$$

Thus $N_e \in L^{L^X}$ is called the fuzzy neighborhood system of e in (X, τ).

Definition 4: Ying (1991) let (X, τ) be a fts, e a fuzzy point in X and A a fuzzy subset of X.

Then the degree to which e is an adherent point of A is given as:

$$ad(e,A) = \inf_{B\subseteq A^c}\left(1 - N_e(B)\right)$$

where, A^c is the complement of A.

Definition 5: Ramadan (1992) A smooth topological space (sts, for short) is an ordered pair (X, τ), where X is a non-empty set and τ: $L^X \to L$ is a mapping satisfying the following properties:

Corresponding Author: A.A. Ramadan, Department of Mathematics, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt

(O1) $\tau(\underline{1}) = \tau(\underline{0}) = 1$

(O2) For all $A_1, A_2, \in L^X, \tau(A_1 \cap A_2) \geq \tau(A_1) \wedge \tau(A_2)$

(O3) $\forall I, \tau(\bigcup_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} \tau(A_i)$

Definition 6: EL Gayyar *et al*. (1994) let (X,τ) be a sts and $\alpha \in L_o$. Then the family: $\tau_\alpha = \{A \in L^X : \tau(A) \geq \alpha\}$. which is clearly a fuzzy topology Chang (1968) sense.

Definition 7**:** Demirici (1997) Let (X,τ) be a sts and $A \in L^X$. Then the τ-smooth interior of A, denoted by:

$$A^0 = \bigcup\{B \in L^X : \tau(B) > 0, B \subseteq A\}$$

Remark 1: Demirici (1997) let τ be a Chang's fuzzy topology (CFT, for short) on the non-empty set X. Then the smooth topology and smooth cotopology τ_s, τ_s^*: $L^X \rightarrow L$, defined by:

$$\tau(A) = \begin{cases} 1, & \text{if } A \in \tau \\ 0, & \text{if } A \notin \tau \end{cases}$$

and $\tau_s^*(A) = \tau(A^c)$ for each $A \in L^X$, identify the CFT τ and corresponding fuzzy cotopology for it. Thus the τ_s-smooth interior of A is:

$$A^0 = \bigcup\{B \in L^X : \tau_s(B) > 0, B \subseteq A\}$$
$$= \bigcup\{B \in L^X : B \in \tau, B \subseteq A\}$$

This show that A^o is exactly the interior of A with respect to τ in Chang (1968) sense.

Lemma 1: Ramadan (1992) $\sup_{\alpha\in L} \sup\{A(x) \wedge B(x): A(x) \geq \alpha\} = \sup_{\alpha\in L} \sup\{\alpha \wedge B(x): A(x) \geq \alpha\}$.

Smooth neighborhood systems of a fuzzy set: Here, we build a smooth neighborhood systems of a fuzzy set in a sts and we give some of its properties.

For a mapping M: $L^X \rightarrow L^{LX}$ and $A \in L^X$, $\alpha \in [0; 1)$; let us define the family $M_A^\alpha = \{B \in L^X: M_A(B) > \alpha\}$; which will play an important role in this part.

Definition 8: Let (X,τ) be a sts and $A \in L^X$: Then a mapping $N_A : L^X \rightarrow L^{L^X}$ is called the smooth neighborhood (nbd, for short) of A with respect to the st τ iff for each $\alpha \in [0,1)$:

$$N_A^\alpha = \{B \in L^X : (\exists C \in \tau^\alpha)(A \subseteq C \subseteq B)\}$$

where, $\tau^\alpha = \{A \in L^X: \tau(A) > \alpha\}$ the strong α- level of τ.

Remark 2:

- The real number $N_A(B)$ is called the degree of nbdness of the fuzzy set B to the fuzzy set A. If the smooth nbd system of a fuzzy set A has the following property: $N_A(L^X) \subseteq \{0, 1\}$, then N_A is called the fuzzy nbd system of A
- We say that the family $(N_A)_\alpha = \{B: N_A(B) > \alpha\}$ is a fuzzy nbd system of A for each $\alpha \in [0,1)$ and $(N_A)_\alpha$ is called the strong α -level fuzzy nbd of A

Proposition 1: Let (X, τ) be a sts and $A \in L^X$. Then a mapping N_A: $L^X \rightarrow L^{LX}$ is the smooth nbd system of A with respect to the st τ iff:

$$N_A(B) \begin{cases} \sup\{\tau(C) : A \subseteq C \subseteq B\}, & \text{if } A \subseteq B \\ 0, & \text{if } A \not\subseteq B \end{cases}$$

Proof:

(1) Suppose that the mapping N_A: $L^X \rightarrow L^{LX}$ is the smooth nbd systems of A with respect to the st τ. Consider the following two cases:

- For the case $A \not\subset B$, suppose that $N_A(B) > 0$. From Definition 1, there exists $C \in \tau^\alpha$ such that $A \subseteq C \subseteq B$, i.e., $A \subseteq B$, a contradiction. Thus $N_A(B) = 0$
- For the case $A \subseteq B$. We may have $N_A(B) = 0$ or $N_A(B) > 0$. If $N_A(B) = 0$, then it is obvious that $N_A(B) = 0 \leq \sup\{\tau (C): A \subseteq C \subseteq B\}$, if $\sup\{\tau (C): A \subseteq C \subseteq B\} = \lambda > 0$, then $\exists C \in L^X$ such that $\tau (C) > 0$ and $A \subseteq C \subseteq B$: We obtain $N_A(B) > 0$, a contradiction

Therefore:

$$N_A(B) = 0 = \sup\{\tau(C) : A \subseteq C \subseteq B\}$$

Now suppose that $N_A(B) = \lambda > 0$. For an arbitrary $0 < \varepsilon \leq \lambda$, we have $N_A(B) = \lambda$-ε, i.e., $B \in N_A^{\lambda-\varepsilon}$. Since the mapping: N_A: $L^X \rightarrow L^{LX}$ is a smooth nbd system of A, $\exists C \in L^X$ such that $C \in \tau^{\lambda-\varepsilon}$ and $A \subseteq C \subseteq B$, i.e., $\sup\{\tau(C): A \subseteq C \subseteq B\} > \lambda$-$\varepsilon$. Since $\varepsilon > 0$ is arbitrary we have:

$$\sup\{\tau(C) : A \subseteq C \subseteq B\} \geq \lambda = N_A(B)$$

On the other hand, let $\sup\{\tau (C): A \subseteq C \subseteq B\} = \gamma > 0$. Then for every $0 < \varepsilon \leq \gamma$, $\exists C \in L^X$ such that $\tau(C) > \gamma$-ε and $A \subseteq C \subseteq B$. Therefore $B \in N_A^{\gamma-\varepsilon}$, i.e., $N_A(B) > \gamma$-ε. Since ε is an arbitrary we have:

$$N_A(B) \geq \gamma = \sup\{\tau(C) : A \subseteq C \subseteq B\}$$

Hence the inequality follows:

(2) For $\alpha\in$ [0, 1), let $B \in N_A^\alpha$, i.e., $N_A(B)>\alpha$: Then we can write $\alpha < N_A(B)= \sup\{\tau (C): A\subseteq C\subseteq B\}$,i.e., $\exists C\in L^X$ such that $\tau(C)>\alpha$, $A \subseteq C \subseteq B$. Then we have:

$$N_A^\alpha \subseteq \{B\in L^X : (\exists C\in \tau^\alpha)(A \subseteq C \subseteq B)\}$$

By the same way we can show that:

$$\{B\in L^X : (\exists C\in \tau^\alpha)(A \subseteq C \subseteq B)\} \subseteq N_A^\alpha$$

Hence:

$$N_A^\alpha = \{B\in L^X : (\exists C\in \tau^\alpha)(A \subseteq C \subseteq B)\}$$

Remark 3: In Proposition 3, the fuzzy subsets A of X can be replaced by the fuzzy points on X, that is, by the special fuzzy subsets e, in this case:

$$N_e(A)\begin{cases} \sup\{\tau(C) : e\in C \subseteq A\}, & \text{if } e\in A \\ 0, & \text{if } e\notin A \end{cases}$$

Proposition 2: Let (X,τ) be a sts and $A\in L^X$. If the mapping $N_A : L^X \to L^{L^X}$ is the smooth nbd system of A with respect to the st τ, then the following properties hold:

(N1) $N_{\underline{0}}(\underline{0}) = N_{\underline{1}}(\underline{1}) = 1$ and $N_A(B) > 0 \Rightarrow A \subseteq B$

(N2) If $A_1 \subseteq A$ and $B \subseteq B_1$, then $N_A(B) \leq N_{A_1}(B_1)$

(N3) $N_A(B_1) \wedge N_A(B_2) \leq N_A(B_1 \cap B_2)$

(N4) $N_A(B) \leq \sup_{A\subseteq C\subseteq B}\{N_A(C) \wedge N_C(B)\}, \forall A,B,C\in L^X$

Proof: (N1) and (N2) follows directly from Definition 1 and Proposition 3. (N3) Suppose that $N_A(B_1) = \alpha_1>0$ and $N_A(B_2)>\alpha_2>0$. Then for a fixed $\varepsilon>0$ such that: $\varepsilon\leq \alpha_1\wedge\alpha_2 \Rightarrow N_A(B_1) > \alpha_1-\varepsilon \geq 0$ and $N_A(B_2) >\alpha_2-\varepsilon\geq 0$. From Definition 1, it is clear that there exists $C_1, C_2\in L^X$ such that:

$$\tau(C_1) > \alpha_1 - \varepsilon, \tau(C_2) > \alpha_2 - \varepsilon \text{ and}$$
$$A \subseteq C_1 \subseteq B_1, A \subseteq C_2 \subseteq B_2$$

Therefore, $\tau(C_1\cap C_2) \geq \tau(C_1)\wedge\tau(C_2) > (\alpha_1-\varepsilon)\wedge(\alpha_2-\varepsilon) = (\alpha_1\wedge\alpha_2)-\varepsilon$ and $A\subseteq C_1\cap C_2 \subseteq B_1\cap B_2$. Thus $N_A(B_1\cap B_2)\geq(\alpha_1\wedge\alpha_2)-\varepsilon$: Since ε is arbitrary, we find that $N_A(B_1\cap B_2) \geq N_A(B_1)\wedge N_A(B_2)$. (N4) $N_A(B)= \sup\{\tau(C): A\subseteq C\subseteq B\}$. From Proposition 3, we obtain $\tau(C)\leq N_A(C)$ and $\tau(C)\leq N_C(B)$.

Thus, $\sup\{\tau (C): A\subseteq C\subseteq B\}\leq \sup\{N_A(C)\wedge N_C(B)\}$. Hence:

$$N_A(B) \leq \sup_{A\subseteq C\subseteq B}\{N_A(C) \wedge N_C(B)\}$$

Smooth neighborhood systems of a fuzzy points:
Definition 9: Let (X,τ) be a sts, e a fuzzy point in X and A be a fuzzy subset of X.

Then the degree to which A is a NBD of e is defined by:

$$N_e(A) = \begin{cases} \sup_{B\subseteq A}\{m(e,B) \wedge \tau(B) : \tau(B) > 0\}, & \text{if } m(e,A) > 0 \\ 0, & \text{otherwise} \end{cases}$$

Thus $N_e \in L^{L^X}$ is called the smooth NBD system of e in (X, τ).

Remark 4: It is clear that when a fuzzy point $e\in B\in L^X$, then $m(e, B) = 1$ and

$$N_e(A) = \begin{cases} \sup_{B\subseteq A}\{\tau(B) : e\in B \subseteq A\}, & \text{if } e\in A \\ 0, & \text{if } e\notin A \end{cases}$$

is the NBD systems in the sense of Demirici (1997)

Remark 5: For any crisp point x in X, we have:

$$N_x(A) = \sup_{B\subseteq A}\{B(x) \wedge \tau(B) : \tau(B) > 0\}, B(x) \neq 0.$$

Proposition 3: The NBD systems N_e of e in sts can be constructed from the cuts τ_α, $\alpha\in(0,1]$, by using the equality:

$$N_e(A) = \sup_{\alpha>0}\{[N_e^*(A)]^\alpha \wedge \alpha\}$$

where, $[N_e^*(A)]^\alpha = \sup\{m(e,B) : B \subseteq A, B\in \tau_\alpha\}$, is the NBD systems in the sense of (Ying, 1994; Theorem 1).

Proof: By using Definition 9, we have:

$$\begin{aligned} N_e(A) &= \sup_{B\subseteq A}\{m(e,B) \wedge \tau(B) : \tau(B) > 0\} \\ &= \sup_{\alpha>0}\sup_{B\subseteq A}\{m(e,B) \wedge \alpha : \tau(B) \geq \alpha\} \\ &= \sup_{\alpha>0}\{\sup_{B\subseteq A}\{B(x) : \tau(B) \geq \alpha\} \wedge \alpha\} \\ &= \sup_{\alpha>0}\{\sup_{B\subseteq A}\{m(e,B) : B\in \tau_\alpha\} \wedge \alpha\} \\ &= \sup_{\alpha>0}\{[N_e^*(A)]^\alpha \wedge \alpha\} \end{aligned}$$

Remark 6: For any crisp point x in X; we have:

$$N_x(A)=\sup_{\alpha>0}\{[N_x^*(A)]^\alpha\wedge\alpha\}$$

where, $[N_x^*(A)]^\alpha=\sup_{B\subseteq A}\{B(x):B\in\tau_\alpha\}$.

Theorem 1: Let (X, τ) be a sts and e a fuzzy point of X. If the mapping N_e: $L^X\to L$ is the smooth NBD systems of e with respect to τ, then the following properties hold:

$$(N1)\,N_e(A)\le m(e,B)$$
$$(N2)\,\text{If } A\subseteq B \text{ and } A,B\in L^x, \text{then } N_e(A)\le N_e(B)$$
$$(N3)\,\text{For all } A,B\in L^x, N_e(A\cap B)\ge N_e(A)\wedge N_e(B)$$
$$(N4)\,N_e(A)\le\sup_{\alpha>0}\{\sup_{B\subseteq A}\{[N_e^*(B)^\alpha\wedge\alpha: \text{for all fuzzy point } d, m(d,B)\le N_d^*(B)\}\}$$

Proof: (N1) and (N2) follows directly from Remark 2.
(N3):

$$\begin{aligned}
&\min(N_e(A),N_e(B))\\
&=\min(\sup_{C\subseteq A}\{m(e,C)\wedge\tau(C):\tau(C)>0\},\sup_{D\subseteq B}\\
&\{m(e,D)\wedge\tau(D):\tau(D)>0\})\\
&=\sup_{D\subseteq B}\sup_{C\subseteq A}\{\min(m(e,C),m(e,D))\wedge\tau(C)\wedge\tau(D):\\
&\tau(C),\tau(D)>0\}\\
&=\sup_{D\subseteq B}\sup_{C\subseteq A}\{\min(m(e,C\cap D)\wedge\tau(C)\wedge\tau(D):\tau(C),\\
&\tau(D)>0\}\\
&\le\sup_{D\subseteq B}\sup_{C\subseteq A}\{\min(m(e,C\cap D)\wedge\tau(C\cap D):\\
&\tau(C),\tau(D)>0\}\\
&\le\sup_{E\subseteq A\cap B}\{(m(e,E)\wedge\tau(E):\tau(E)>0,C\cap D=E\}\\
&=N_e(A\cap B)
\end{aligned}$$

(N4) Combining axiom (4) in Theorem 1, in (Ying, 1994) and Proposition 4, (N4) follows.

Theorem 2: Let the mapping N_e: $L^X\to L$ satisfy the conditions (N1)-(N4), then the mapping τ: $L^X\to L$ defined by:

$$\tau(A)=\begin{cases}\inf_e\{m(e,A)\wedge N_e(A)\}, & \text{if } m(e,A)>0, A\ne\underline{0} \text{ and } A\ne\underline{1}\\ 1, & \text{if } A=\underline{0} \text{ or } A=\underline{1}\end{cases}$$

Where:
$A\in L^X$ = A st on X, furthermore the mapping
N_e = Exactly the smooth nbd systems of e with respect to st τ.

Proof: (O1) Obvious.

(O2):

$$\begin{aligned}
\tau(A\cap B)&=\inf_e\{m(e,A\cap B)\wedge N_e(A\cap B)\}\\
&\ge\inf_e\{m(e,A\cap B)\wedge(\min(N_e(A),N_e(B)))\}\\
&=\inf_e\{\min(m(e,A),m(e,B))\wedge(\min(N_e(A),N_e(B)))\}\\
&=\min\{\inf_e\{m(e,A)\wedge N_e(A)\},\inf_e\{m(e,B)\wedge N_e(B)\})\\
&=\min(\tau(A),\tau(B))
\end{aligned}$$

(O3):

$$\begin{aligned}
\tau(\textstyle\bigcup_{i\in I}A_i)&=\inf_e\{m(e,\textstyle\bigcup_{i\in I}A_i)\wedge N_e(\textstyle\bigcup_{i\in I}A_i)\}\\
&=\inf_e\{\sup_{i\in I}m(e,A_i)\wedge N_e(\textstyle\bigcup_{i\in I}A_i)\}\\
&\ge\inf_e\{\sup_{i\in I}m(e,A_i)\wedge\inf_{i\in I}N_e(A_i)\}\\
&\ge\inf_e\{\inf_{i\in I}m(e,A_i)\wedge\inf_{i\in I}N_e(A_i)\}\\
&=\inf_{i\in I}\inf_e\{m(e,A_i)\wedge N_e(A_i)\}\\
&=\inf_{i\in I}\tau(A_i)
\end{aligned}$$

Now, we show that the mapping N_e: $L^X\to L$ which satisfies the conditions (N1)-(N4) is exactly the smooth NBD systems of e for the sts (X, τ): Let the mapping M_e: $L^X\to L$ be the smooth NBD systems of e of the sts (X,τ). Then applying (N_1) we have:

$$\begin{aligned}
M_e(A)&=\sup_{B\subseteq A}\{m(e,B)\wedge\tau(B)\}\\
&=\sup_{B\subseteq A}\{m(e,B)\wedge\inf_p(m(p,B)\wedge N_p(B))\}
\end{aligned}$$

Since:

$$\inf_p(m(p,B)\wedge N_p(B))=\inf_p m(p,B)\wedge\inf_p N_p(B)\le\inf_p N_p(B)\le N_e(B)$$

Thus:

$$M_e(A)\le\sup_{B\subseteq A}\{m(e,B)\wedge N_e(B)\}\le N_e(A) \qquad (1)$$

On the other hand, using (N4) and Theorem 1, in (Ying, 1994) we may write:

$$\begin{aligned}
&N_{x_\lambda}(A)\le\sup_{\alpha>0}\{\sup_{B\subseteq A}\{[N_{x_\lambda}^*(B)]^\alpha\wedge\alpha:\text{for}\\
&\text{each fuzzy point } d, m(e,B)\le N_d^*(B)\}]\\
&\le\sup_{\alpha>0}\{\sup_{B\subseteq A}\{[N_{x_\lambda}^*(B)]^\alpha\wedge\alpha:\text{for each}\\
&\text{crisp point } y, m(y,B)\le N_y^*(B)\}]\\
&=\sup_{\alpha>0}\{\sup_{B\subseteq A}\{[N_{x_\lambda}^*(B)]^\alpha\wedge\alpha:B\in\tau_\alpha\}\}\\
&=\sup_{\alpha>0}\{\sup_{B\subseteq A}\{\min(1,1-\lambda+N_x^*(B))\wedge\alpha:\\
&B\in\tau_\alpha\}\} \qquad (2)\\
&\le\sup_{\alpha>0}\{\sup_{B\subseteq A}\{\min(1,1-\lambda+m(x,B))\wedge\alpha:\\
&B\in\tau_\alpha\}\}\\
&=\sup_{\alpha>0}\{\sup_{B\subseteq A}\{m(x_\lambda,B)\wedge\alpha:B\in\tau_\alpha\}\}\\
&=\sup_{B\subseteq A}\{m(x_\lambda,B)\wedge\tau(B)\}\\
&=M_{x_\lambda}(A)
\end{aligned}$$

Hence, the equality $N_e = M_e$ follows at once from (1) and (2).

Definition 10: Let (X,τ) be a sts, e a fuzzy point in X and A a fuzzy subset of X. Then the degree to which e is an adherent point of A is given as:

$$ad(e,A) = \inf_{B\subseteq A^c}(1 - N_e(B)$$

where, A^c is the complement of A.

Remark 6: For any crisp point x in X, we have:

$$ad(x,A) = \inf_{B\subseteq A^c}(1 - N_x(B))$$

Proposition 4:

$$ad(e,A) = \inf_{\alpha>0}\{[ad(e,A)]^\alpha \vee (1-\alpha)\}$$
$$[ad(e,A)]^\alpha = \inf_{B\subseteq A^c}(1-[N_e^*(B)]^\alpha)$$

Proof: Follows from Proposition 4.

Proposition 5:

$$N_{x_\lambda}(A) \le \sup_{\alpha>0}\{\min(1,1-\lambda+[N_x^*(A)^\alpha]) \wedge \alpha)\}$$

Proof:

$$\begin{aligned} N_{x_\lambda}(A) &= \sup_{\alpha>0}\{[N^*_{x_\lambda}(A)]^\alpha \wedge \alpha\} \\ &= \sup_{\alpha>0}\{\sup_{B\subseteq A}\{m(x_\lambda,B) : B\in \tau_\alpha\} \wedge \alpha\} \\ &= \sup_{\alpha>0}\{\sup_{B\subseteq A}\{\min(1,1-\lambda+m(x,B)) : B\in \tau_\alpha\} \wedge \alpha\} \\ &\le \sup_{\alpha>0}\{\min(1,1-\lambda+\sup_{B\subseteq A}\{m(x,B) : B\in \tau_\alpha\}) \wedge \alpha\} \\ &= \sup_{\alpha>0}\{\min(1,1-\lambda+[N^*_x,(A)]^\alpha) \wedge \alpha\} \end{aligned}$$

Fuzzy smooth r-neighborhood:

Definition 11: Let (X,τ) be a sts, $A\in L^X$, e a fuzzy point in X and $r\in L_0$. Then the degree to which A is a fuzzy smooth r-nbd system of e is defined by:

$$N_e(A,r) = \sup_{B\subseteq A}\{m(e,B) : \tau(B) \ge r\}$$

A mapping N_e: $L^X \times L_0 \to L$ is called the fuzzy smooth r-nbd system of e.

Theorem 2: Let (X,τ) be a sts and N_e the fuzzy smooth r-nbd system of e. For A, $B\in L^X$ and r, $s\in L_0$, it satisfies the following properties:

(1) $N_e(A,r) \le m(e,A)$ for each $r\in L_0$
(2) $N_e(A, r) \le N_e(B, r)$, if $A\subseteq B$
(3) $N_e(A,r) \wedge N_e(B,r) \le N_e(A\cap B,r)$
(4) $N_e(A, r) \le \sup\{N_e(B, r): B \le A,\ m(d, B) \le N_d(B, r)$; for all fuzzy point d in X$\}$
(5) $N_e(A, r) \ge N_e(A, s)$, if $r \le s$
(6) $N_{x_t}(A,r) = \min(1,1-t+N_{x_1}(A,r))$

Proof: (2) and (5) are easily proved.

(1) It is proved from the following:

$$\begin{aligned} N_e(A,r) &= \sup\{m(e,B_i) : B_i \le A, \tau(B_i) \ge r\} \\ &= \{m(e,\cup B_i) : \cup B_i \le A, \tau(\cup B_i) \ge r\} \\ &\le m(e,A) \end{aligned}$$

Suppose there exist A, $B\in L^X$ and $r\in L_0$ such that:

$$N_e(A,r) \wedge N_e(B,r) > t > N_e(A\cap B,r)$$

Since $N_e(A,r)>t$ and $N_e(B,r)>t$, there exist $C_1,C_2\in L^X$ with:

$$C_1 \subseteq A, \tau(C_1) \ge r, C_2 \subseteq B, \tau(C_2) \ge r$$

Such that:

$$m(e,C_1) \wedge m(e,C_2) = m(e,C_1 \cap C_2) > t$$

On the other hand, since:

$$C_1 \cap C_2 \subseteq A\cap B, \tau(C_1\cap C_2) \ge r$$

We have:

$$N_e(A\cap B,r) \ge m(e,C_1\cap C_2) > t$$

It is a contradiction.

(4) If $\tau(B)\ge r$, then $N_d(B,r) = m(d,B)$; for each fuzzy point d in X. It implies:

$$\begin{aligned} &N_e(A,r) \\ &= \sup\{m(e,B) : B\subseteq A, \tau(B) \ge r\} \\ &= \sup\{N_e(B,r) : B\subseteq A, N_d(B,r) = m(d,B), \text{for all fuzzy point d in X}\} \\ &\le \sup\{N_e(B,r) : B\subseteq A, m(d,B) \le N_d(B,r), \text{for all fuzzy point d in X}\} \end{aligned}$$

(6) It proved from:

$$N_{x_t}(A,r) = \sup\{m(x_t,B): B \subseteq A, \tau(B) \geq r\}$$
$$= \sup\{\min(1, 1-t+B(x)): B \subseteq A, \tau(B) \geq r\}$$
$$= \min(1, 1-t+\sup\{B(x): B \subseteq A, \tau(B) \geq r\})$$
$$= \min(1, 1-t+N_x(A,r))$$

Theorem 3: Let N_e be the fuzzy smooth r-nbd system of e satisfying the above conditions (1)-(5), the function $\tau_N : L^X \to L$ defined by:

$$\tau_N(A) = \vee\{r \in L_0 : m(e,A) = N_e(A,r) \text{ for all fuzzy point e in X}\}$$

has the following properties:

(1) τ_N is a st. on X
(2) If N_e is the fuzzy nbd systems of e induced by (X,τ), then $\tau_N = \tau$
(3) If N_e satisfy the conditions (1)-(6), then:

$$\tau_N(A) = \vee\{r \in L_0 : m(x,A) = N_x(A,r), \forall x \in X\}$$

Proof: (1) We will show that $\tau_N(B_1 \cap B_2) \geq \tau_N(B_1) \wedge \tau_N(B_2)$, for any $B_1, B_2 \in L^X$.

Suppose there exist $B_1, B_2 \in L^X$ and $r \in L_0$ such that:

$$\tau_N(B_1 \cap B_2) < r < \tau_N(B_1) \wedge \tau_N(B_2) \qquad \text{(I)}$$

For each $i \in \{1,2\}$ there exists $r_i \in L_0$ with:

$m(e, B_i) = N_e(B_i, r_i)$; for all fuzzy point e in X (II)

Such that: $\tau_N(B_i) \geq r_i > r$,

From (I), (II) and (5), we have:

$$m(e,B_i) = N_e(B_i, r_i) \leq N_e(B_i r) \leq m(e,B_i)$$

It implies $m(e, B_i) = N_e(B_i, r)$: Furthermore:

$$m(e, B_1 \cap B_2) = N_e(B_1, r) \wedge N_e(B_2, r)$$
$$\leq N_e(B_1 \cap B_2, r)$$
$$\leq m(e, B_1 \cap B_2)$$

Thus, $N_e(B_1 \cap B_2, r) = m(e, B_1 \cap B_2)$, i.e., $\tau N(B_1 \cap B_2) \geq r$. It is a contradiction for the Eq. I.

Suppose there exists $B = \cup_{i\in\Gamma} B_i \in L_X$ and $r_0 \in L_0$ such that:

$$\tau_N(B) < r_0 < \bigwedge_{i\in\Gamma} \tau_N(B_i) \qquad \text{(III)}$$

For each $i \in \Gamma$, there exists $r_i \in L_0$ with

$m(e, B_i) = N_e(B_i, r_i)$; for all fuzzy point e in X (IV)

Such that: $\tau_N(B_i) \geq r_i > r$

From (I), (IV) and (5), we have:

$$m(e,B_i) = N_e(B_i, r_i) \leq N_e(B_i, r) \leq m(e,B_i)$$

It implies $m(e, B_i) = N_e(B_i, r)$: Furthermore:

$$m(e, \cup_{i\in\Gamma} B_i) = \vee_{i\in\Gamma} m(e,B_i)$$
$$= \vee_{i\in\Gamma} N_e(B_i, r_i)$$
$$\leq N_e(\cup_{i\in\Gamma} B_i, r)$$
$$\leq m(e, \cup_{i\in\Gamma} B_i).$$

Thus, $N_e(\cup_{i\in\Gamma} B_i, r) = m(e, \cup_{i\in\Gamma} B_i)$, i.e., $\tau_N(\cup_{i\in\Gamma} B_i) \geq r_0$. It is a contradiction for the Eq. III.

(2) Suppose there exists $A \in L^X$ such that:

$$\tau_N(A) > \tau(A)$$

From the Definition of τ_N, there exists $r_0 \in L_0$ with $m(e, A) = N_e(A, r_0)$ such that:

$$\tau_N(A) \geq r_0 > \tau(A)$$

Since:

$$m(e,A) = N_e(A, r_0) = \sup\{m(e,B_i): B_i \subseteq A, \tau(B_i) \geq r_0\}$$

Then, for each $x \in X$:

$$(\cup B_i)(x) = \sup\{m(x,B_i): B_i \subseteq A\} = m(x,A) = A(x)$$

Thus, $A = \cup B_i$. So, $\tau(A) \geq r_0$. It is a contradiction.

Suppose there exists $A \in L^X$ such that:

$$\tau_N(A) < \tau(A)$$

There exists $r_1 \in L_0$ such that:

$$\tau_N(A) < r_1 \leq \tau(A)$$

Since $\tau(A) \geq r_1$, we have:

$$N_e(A, r_1) = \sup\{m(e,B): B \subseteq A, \tau(B) \geq r_1\} = m(e,A)$$

Hence $\tau_N(A) \geq r_1$. It is a contradiction.

CONCLUSION

(3) We only show that $m(x_t, A) = Nx_t(A, r)$, for all fuzzy point x_t in X iff $m(x, A) = A(x) = N_x(A, r)$, $\forall x \in X$:

($\Rightarrow$) It is trivial.
($\Leftarrow$) From the condition (6):

$$\begin{aligned} N_{x_t}(A,r) &= \min(1, 1-t+N_x(A,r)) \\ &= \min(1, 1-t+m(x,A)) \\ &= \min(1, 1-t+A(x)) \\ &= m(x_t, A). \end{aligned}$$

Example 1: Let X = {a, b} be a set, N a natural number set and $B \in L^X$ as follows:

$$B(a) = 0.3, \quad B(b) = 0.4$$

We define a smooth fuzzy topology:

$$\tau(A) = \begin{cases} 1, & \text{if } A = \underline{0} \text{ or } \underline{1}, \\ \frac{1}{2}, & \text{if } A = B, \\ 0, & \text{otherwise} \end{cases}$$

From Definition 1, N_a, N_b: $L^X \times L_0 \rightarrow L$ as follows:

$$N_a(A) = \begin{cases} 1, & \text{if } A = \underline{1}, \quad r \in L_0 \\ 0.3, & \text{if } \underline{1} \neq A \supseteq B, \quad 0 < r \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$N_b(A) = \begin{cases} 1, & \text{if } A = \underline{1}, \quad r \in L_0 \\ 0.4, & \text{if } 1 \neq A \supseteq B, \quad 0 < r \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

From Theorem 2 and Theorem 3 (3), we have:

$$\tau_N(A) = \begin{cases} 1, & \text{if } A = \underline{0} \text{ or } \underline{1}, \\ \frac{1}{2}, & \text{if } A = B, \\ 0, & \text{otherwise} \end{cases}$$

REFERENCES

Abdel-Sattar, M.A., 2006. Some structures of L-fuzzy topology. J. Fuzzy Math., 14: 751-766.

Chang, C.L., 1968. Fuzzy topological spaces. J. Math. Anal. Appli., 24: 182-190.

Chattopadhyay, K.C. and S.K. Samanta, 1993. Fuzzy topology: Fuzzy closure operator, fuzzy compactness and fuzzy connectedness. Fuzzy Sets and Syst., 54: 207-212. DOI: 10.1016/0165-0114(93)90277-O

Demirici, M., 1997. Neighborhood structures of smooth topological spaces. Fuzzy Sets Syst., 92: 123-128. DOI: 10.1016/S0165-0114(96)00132-7

El Gayyar, M.K., E.E. Kerre and A.A. Ramadan, 1994. Almost compactness and near compactness in smooth topological spaces. Fuzzy Sets Syst., 62: 193-202. DOI: 10.1016/0165-0114(94)90059-0

Höhle, U. and S.E. Rodabaugh, 1998. Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory. 1st Edn., Springer, USA., ISBN: 10: 0792383885, pp: 736.

Kubiak, T. and A. Šostak, 1997. Lower set-valued fuzzy topologies. Quaest. Math., 20: 423-429. DOI: 10.1080/16073606.1997.9632016

Pu, P.M. and Y.M. Liu, 1980. Fuzzy topology I: Neighborhood structure of a fuzzy point and Moore-Smith convergence. J. Math. Anal. Appli. 76: 571-599.

Ramadan, A.A., 1992. Smooth topological spaces. Fuzzy Sets Syst., 48: 371-375. DOI: 10.1016/0165-0114(92)90352-5

Ramadan, A.A., S.N. El-Deeb and M.A. Abdel-Sattar, 2001. On smooth topological spaces IV. Fuzzy Sets Syst., 119: 473-482. DOI: 10.1016/S0165-0114(99)00083-4

Ramadan, A.A., M.A. Abdel-Sattar and Y.C. Kim, 2009. The fuzzy neighborhood structures. Int. J. Math. Comput., 3: 146-156.

Šostak, A.P., 1985. On a fuzzy topological structures. Suppl. Rend. Circ. Mat Palermo Ser. II, 11: 89-103.

Ying, M., 1991. A new approach for fuzzy topology I. Fuzzy Sets and Syst., 39: 303-321. DOI: 10.1016/0165-0114(91)90100-5

Ying, M., 1994. On the method of neighborhood systems in fuzzy topology. Fuzzy Sets Syst., 68: 227-238. DOI: 10.1016/0165-0114(94)90048-5