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On Hyperplanes of the Geometry $D_{4,2}$ and their Related Codes

Abdelsalam Abou Zayda
Department of Mathematics, Al-Aqsa University, Gaza, Palestine

Abstract: Problem statement: The point-line geometry of type $D_{4,2}$ was introduced and characterized by many authors such as Shult and Buekenhout and in several researches many of geometries were considered to construct good families of codes and this forced us to present very important substructures in such geometry that are hyperplanes. **Approach:** We used the isomorphic classical polar space $\Omega^+(8, F)$ and their combinatorics to construct the hyperplanes and the family of certain codes related to such hyperplanes. **Results:** We proved that each hyperplane is either the set Δ_2 (p) which consisted of all points at a distance mostly 2 from a fixed point p or a Grassmann geometry of type $A_{3,2}$ and then we presented a new family of non linear binary constant-weight codes. **Conclusion:** The hyperplanes of the geometry $D_{4,2}$ allow us to discuss further substructures of the geometry such as veldkamp spaces.

Key words: Hyperplane, grassmann geometry, constant weight code

INTRODUCTION

In[1] Bruyn characterized the hyperplanes of the dual polar spaces DQ (2n,K) and DQ^- (2n+1,K). In[2] and[3] he also determined all hyperplanes of DW 2n-1,q), $q \neq 2$ and DW 5, 2^h). In[4] Bruyn and Pralle presented a classification of all hyperplanes of DH (5,4). Shult[5] proved that every hyperplane of the Half-spin geometry arises from embedding. This study presented a description for two classes of hyperplanes of point-line geometry of type $D_{4,2}$ which was characterized completely in[7].

For the following definitions[6].

A given set I, geometry Γ over I is an ordered triple Γ= (X, D), where X is a set, D is a partition $\{X_i\}$ of X indexed by I, X_i are called components, is a symmetric and reflexive relation on X called incidence relation such that: A point-line geometry (P, L) is simply a geometry for which $|I| = 2$, one of the two types is called points, in this notation the points are the members of Pand the other type is called lines. Lines are the members of L. If $p \in P$ and $l \in L$, then $p * l$ if and only if $p \in l$. In point-line geometry (P, L), it's said that two points of P are collinear if and only if they are incident with a common line.

A subspace of a point-line geometry Γ = (P, L) is a subset $X \subseteq P$ such that any line which has at least two of its incident points in X has all of its incident points in X. A hyperplane of a point line geometry is a proper subspace meets each line in at least one point. $\langle X \rangle$ means the intersection over all subspaces containing X, where $X \subseteq P$. Lines incident with more than two points are called thick lines, but those incident with exactly two points are called thin lines.

$x^\perp$ means a set of all points in P collinear with x, including x itself. A clique of P is a set of points in which every pair of points are collinear. A partial linear space is a point-line geometry (P, L), in which every pair of points are incident with at most one line and all lines have cardinality at least 2. A point line geometry Γ = (P, L) is called singular or (linear) if every pair of points is incident with a unique line.

The singular rank of a space Γ is the maximal number n (possibly ∞) for which there is a chain of distinct subspaces $\emptyset \neq X_0 \subset X_1 \subset ... \subset X_n$ such that X_i is singular for each i, $X_i \neq X_j$, $i \neq j$, for example rank $(\emptyset)$ = -1, rank({p}) = 0 where p is a point and rank(L) = 1 where L a line.

In a point-line geometry Γ = (P, L), a path of length n is a sequence of n+1 $(x_0, x_1,..,x_n)$ where, (x_i,x_{i+1}) are collinear, x_0 is the initial point and x_n is the end point. A geodesic from a point x to a point y is a path of minimal possible length with initial point x and end point y. This length is denoted by d_Γ (x, y). Diameter of Γ is the maximal distance between the points of Γ, i.e., diameter (Γ) = maximum $\{d(x, y), x, y \in \Gamma\}$. A geometry Γ is called connected if and only if for any two of its points are connected by a bath. A subset X of P is said to be convex if X contains all points of all geodesics connecting two points of X.

A gamma space is point-line geometry such that for every point-line pair (p, l), p is collinear with no point, exactly one point, or all points of l, i.e., $p^{\perp}\cap l$ is empty, consists of a single point, or l.

A polar space is a point-line geometry Γ = (P, L) satisfying the Buekenhout-Shult axiom:

For each point-line pair (p, l) with p not incident with l; p is collinear with one or all points of l, that is $|p^{\perp}\cap l| = 1$ or else $p^{\perp} \supset l$. Clearly this axiom is equivalent to saying that $p^{\perp}$ is a geometric hyperplane of Γ for every point p∈ P.

A point-line geometry Γ = (P, L) is called a projective plane only in case if satisfies the following conditions:

- Γ is a linear space; every two distinct points x, y in P lie exactly on one line
- Every two lines intersect in one point
- There are four points no three of them are on a line

A point-line geometry Γ = (P, L) is called a projective space if the following conditions are satisfied:

- Every two points lie exactly on one line
- If l_1, l_2 are two lines $l_1\cap l_2 \neq \emptyset$, then $\langle l_1, l_2\rangle$ is a projective plane. ($\langle l_1, l_2\rangle$ means the smallest subspace of Γ containing l_1 and l_2)

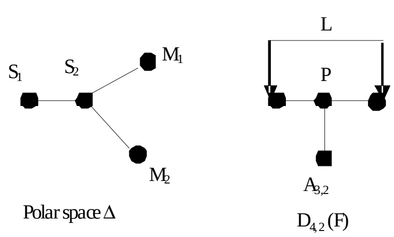
A point-line geometry Γ = (P, L) is called a parapolar space only in case it satisfies the following properties:

- Γ is a connected gamma space
- for every line l, $l^{\perp}$ s not a singular subspace
- for every pair of non-collinear points x, y; $x^{\perp}\cap y^{\perp}$ is either empty, a single point, or a non-degenerate polar space of rank at least 2

If x, y are distinct points in Pand if $|x^{\perp}\cap y^{\perp}| = 1$, then (x, y) is called a special pair and if $x^{\perp}\cap y^{\perp}$ is a polar space, hence (x, y) is called a polar pair (or a symplectic pair). A parapolar space is called a strong parapolar space if it has no special pairs.

MATERIALS AND METHODS

First we present a construction of the geometry $D_{4,2}$:



In[7] the geometry $D_{4,2}$ was introduced to be isomorphic to the classical polar space $\Delta = \Omega^{+}(8, F)$ that comes from a vector space of dimension 8 over a finite field F = GF(q) with a symmetric bilinear form. The set S_1 consists of all totally isotropic 1-dimentional subspaces of the vector space V and S_2 consists of all totally 2-dimensional subspaces of V. The two classes M_1, M_2 consist of maximal totally isotropic 4-dimensional subspaces. Two 4-subspaces fall in the same class if their intersection is of even dimension. Hence the geometry $D_{4,2}$ (F) is a point-line geometry (P, L), whose set of points P is corresponding to the class $S_2$and whose each line is corresponding to the totally isotropic (1, 4)-dimensional subspaces (A, B) and $A\subseteq B$. A point C is incident with a line (A, B) if and only if $A\subset C\subset B$ as a subspaces of V.

To define the colinearity, let C_1 and C_2 be two point (the points are the T.I 2-spaces), then C_1 is collinear to C_2 if and only if the intersection of C_1 and C_2 is a T.I 1-dimensional space, $C_1\cap C_2$ in addition to the complement of C_1 and C_2 must form a T.I 3-dimensional space and then contained in a T.I 4-space. The elements of the class M_2 are corresponding to the class of geometries of type $A_{3,2}$ that are convex polar spaces of rank 2 and then they represent symplecta in the geometry $D_{4,2}$. As a result the symplecta of $D_{4,2}$ (F) are the Grassmannians of type $A_{3,2}$(F) that are corresponding to the collection of TI 4-dimensional spaces.

Notation: Let the map Ψ: P→V be defined as above, i.e., Ψ(p) is the T.I. 2-dimensional subspace corresponding to the point p. We will use Ψ for the rest of the geometry; for example $\Psi(A_{3,2})$ is the T.I. 4-dimensional subspace corresponding to a geometry of type $A_{3,2}$. The inverse map Ψ^{-1} will be used for the inverse; for example Ψ^{-1}(C) is the point corresponding to the T.I. 2-dimensional subspace C.

The following Theorem is a good characterization of the point-line geometry $D_{4,2}$.

Proposition 1: Let Γ = (P, L) be a point-line geometry of type $D_{4,2}$, then the following are satisfied:

- (P_1) Γ is a strong parapolar space of diameter 3
- (P_2) The symplecta of the geometry are of type $A_{3,2}$
- (P_3) If (p, S) is a pair of non-incident point-symplecton, then rank($p^{\perp}\cap$ S)= -1 ,1
- (P_4) If S_1 and S_2 are two different symplecta of $D_{4,2}$, then rank($S_1\cap S_2$)=-1, 0

Proof[7]: For the result that shows we could generate a non linear binary constant-weight code using the second construction of hyperplanes, we present some basic Definitions.

Basic definitions: For the following definitions see[8]. A code C of length n and size M over a field F is just a subset of F^n of cardinality M, then C is (n, M)-code. Thus each code consists of codwords (vectors in F^n) and the number of codwords is the size of the code.

The Hamming weight of $u = (x_1, x_2, \dots, x_n)$ is the number of non-zero coordinates x_i, I = 1, 2, …, n, it is denoted by $w_h(u)$.

Let C be a code of length n and u, v be two codwords. The hamming distance between u and v, $d_h(u, v)$, is the number of coordinate in which they differ, that is $d_h(u, v) = w_h(u+v)$. If $d = \text{minimum}\{d_h(u, v): u, v \in C, u \neq v\}$; then d is called the minimum distance of C, in this case we say that C is (n, M, d)-code. If C is a linear vector subspace of F^n, then C is called a linear code and if the dimension of C is k, we say that C is [n, k, d]-code. If all codwords in C have the same hamming weight w then C is called a constant-weight code. An (n, M, d, w)-code is a constant-weight (n, M, d)-code with w as the common weight of all codwords. If the code C has two weights w_1 and w_2, then C is called a two-weight code (n, M, d, w_1, w_2).

In several studies many of geometries have been considered to construct good families of codes, In[9] the geometric ideas has been used to define a non-linear binary constant-weight code Shult sets in different point-line geometry. In this study we present a new construction for a binary constant weight code using the class of Grassmann geometry $A_{3,2}$ that are isomorphic to the maximal TI 4-spaces in the classical polar space $\Omega^+(8, F)$.

Propositions 2, 3 and 4 is used to prove the result at Theorem. The propositions and their proofs can be found in[10].

Proposition 2[10]: The number of subspaces of dimension k in a vector space of dimension n over GF(q) is:

$$\frac{(q^n-1)(q^n-q)\ \dots\ (q^n-q^{k-1})}{(q^k-1)(q^k-q)\ \dots\ (q^k-q^{k-1})}$$

Proof: This is the proof of Proposition 1.4.1 in[10].

Remark: This number in Proposition 2 is called a Gaussian coefficient and is denoted by:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q$$

Proposition 3[10]: Let V be equipped with a bilinear form then the number of totally isotropic k-subspaces is the following:

In the symplectic case W(2n,q):

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{i=0}^{k-1}(q^{n-i}+1)$$

In the orthogonal case $\Omega(2n+1, q)$:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{i=0}^{k-1}(q^{n-i}+1)$$

In the hyperbolic case $\Omega^+(2n, q)$:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{i=0}^{k-1}(q^{n-i-1}+1)$$

In the elliptic case $\Omega^-(2n+2, q)$:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{i=0}^{k-1}(q^{n-i-1}+1)$$

Proof [10]:

Proposition 4[10]: The numbers of maximal totally singular subspaces of the finite classical polar spaces are given by the following formulae:

$$|\Sigma(W_{2n}(q))| = (q+1)(q^2+1)\ \dots\ (q^{2n}+1)$$
$$|\Sigma(\Omega(2n+1,q))| = (q+1)(q^2+1)\ \dots\ (q^{n+1}+1)$$
$$|\Sigma(\Omega^+(2n,q))| = 2(q+1)(q^2+1)\ \dots\ (q^n+1)$$
$$|\Sigma(\Omega^-(2n,q))| = (q^2+1)(q^3+1)\ \dots\ (q^n+1)$$
$$|\Sigma(H^+(2n,q^2))| = (q+1)(q^3+1)\ \dots\ (q^n+1)$$

RESULTS AND DISCUSSION

The following result shows that the set $\Delta_2(p)$ forms a construction of hyperplanes of the geometry $D_{4,2}$.

Theorem 1: For a fixed point p, The set $\Delta_2(p)$ forms a hyperplane of the geometry $D_{4,2}$. Where $\Delta_2(p)$ is the set of all points that are of distance at most 2 from the fixed point, namely $\Delta_2(p) = \{x \in P: d(x, p) \leq 2\}$.

Proof: Let l be a line in $D_{4,2}(q)$ that is identified by the two points r and s such that $\Psi(r) = \langle x_1, x_3\rangle$ and $\Psi(s) = \langle x_2, x_3\rangle$ and let $\Psi(p) = \langle y_1, y_2\rangle$. Now if $\Psi(p) \subset \Psi(l)$, then $p \in l$ and $p \in \Delta_2(p)$ (because d(p, p) = 0), so $l \cap \Delta_2(p) \neq \varphi$.

If $\Psi(p)$ is not contained in $\Psi(l)$, then there are two cases:

- $\Psi(l) \cap \Psi(p)$ = 1-sapace = <x>, $x = x_3 = y_2$. If $y_1^{\perp} \cap \Psi(l)$ = <x, x_1, x_2>, then <y_1, x, x_2> forms a TI 3-space and p is collinear to s. This means that s $\in \Delta^*_2(p)$, then $l \cap \Delta_2(p) \neq \varphi$
- $\Psi(l) \cap \Psi(p)$ = 0-sapace. If $y_1^{\perp} \cap \Psi(l)$ = <x_3, x_1, x_2> and $y_2^{\perp} \cap \Psi(l)$ = <x_3, x_1, u>, then there is a point q such that $\Psi(q)$ = <x_3, y_2>. Now since <y_1, y_2, x_3> is a TI 3-space, the point q is collinear to the point sand since <y_2, x_3, x_2> form TI 3-spaces, the point q is collinear to the point p which means d (p, s) = 2. Then s $\in \Delta_2(p)$, so $l \cap \Delta_2(p) \neq \varphi$. Then $\Delta_2(p)$ is a hyperplane of the geometry $D_{4,2}$

The following result shows that the Grassmann geometries of type $A_{3,2}$ form the second construction of hyperplanes of such geometry.

Theorem2 in $D_{4,2}$ the class of Grassmann geometries of type $A_{3,2}$ form hyperplanes in the geometry,
Proof: Let l be a line with two point p and q , where $\Psi(p)$ = <x_1, x>, $\Psi(q)$ = <x_2, x> and $\Psi(l)$ = (<x>, <x_1, x, x_2, u>). Let D be one of the Grassmann geometry of type $A_{3,2}$ in $D_{4,2}$. Then $\Psi(D)$ = TI 4-space = <y_1, y_2, y_3, y_4>. Now we prove that $l \cap D \neq \varphi$. Referring to the construction of $D_{4,2}$, $\Psi(D)$ and $\Psi(l)$ form TI 4-spaces belong to different classes of the set of all TI 4-spaces, which means that $\Psi(l) \cap \Psi(D)$ has odd dimensions. Then for $\Psi(l) \cap \Psi(D)$ in just two cases:

- $\Psi(l) \cap \Psi(D)$ = 1-space = <y>, where $y = y_4 = x$. Then $y_1^{\perp} \cap \Psi(l)$ = <x_2, x_1, y>. There is a point $r \in D$ such that $\Psi(r)$ = <y_1, y> and <y_1, y, x_2, x_1> forms a TI 4-spacecontains both $\Psi(p)$ and $\Psi(q)$. So, $\Psi(r) \subset \Psi(l)$ and $r \in l$. Then $l \cap D = \{r\}$
- $\Psi(l) \cap \Psi(D)$ = 3-space. In this case <x, x_1, x_2> $\subset \Psi(D)$. Then $\Psi(p)$ = <x, x_1> $\subset \Psi(D)$,so, $p \in D$ and $l \cap D \neq \varphi$, this complete the proof

Now the following result presents a new family of binary non linear constant-weight codes by using the second construction of hyperplanes of the geometry $D_{4,2}$ (Grassmann geometry of type $A_{3,2}$).

Theorem 3: Let Γ be the point-line geometry $D_{4,2}$ and H be the class of the second construction of hyperplanes of Γ. Let $G = (g_{ij})$ be the incidence matrix, where:

$$g_{ij} = \begin{cases} 1 & \text{if} \quad p_i \in H \\ 0 & \text{if} \quad p_i \notin H \end{cases}$$

Then the rows of G represent a binary constant-weight (n, M, d, w) non-linear code of parameters: $n = |P| = (q^2+1)^2(q^3+1)(q^2+q+1)$.

$M = |H|$ = (the number of hyperplanes of Γ) = $(q+1)(q^2+1)(q^3+1)(q^4+1)$, $w = |H|$ (H is a hyperplane of Γ) = $(q^2+q+1)(q^2+1)$, d = 2w-1.

Proof: The columns of the incidence matrix represent the points of the geometry $D_{4,2}$. Then n represents the number of point of Γ. The points of Γ correspond to all TI 2-spaces in the classical polar space $\Omega^+(8, q)$. Then the number of points n is equal to the number of TI 2-spaces in $\Omega^+(8, q)$ which can be given by:

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \frac{(q^4-q)(q^4-1)}{(q^2-q)(q^2-1)}$$

$$\prod_{i=0}^{1}(q^{3-i}+1) = (q^3+1)(q^2+1)$$

Then:

$$n = |P| = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q \prod_{i=0}^{1}(q^{3-i}+1) = \frac{(q^4-q)(q^4-1)(q^3+1)(q^2+1)}{(q^2-q)(q^2-1)}$$

$$n = (q^2+1)^2(q^3+1)(q^2+q+1)$$

Now the rows of the incidence matrix represent the hyperplanes of the geometry. Since the hyperplanes correspond to the maximal TI spaces in the classical polar space $\Omega^+(8, q)$, the number of maximal TI spaces which is equal to the size of the code M is evaluated. The number of maximal TI 4-spaces is:

$$|\Sigma(\Omega^+(8,q))| = (q+1)(q^2+1)(q^3+1)(q^4+1)$$

Then the size of the code equals:

$$M = (q+1)(q^2+1)(q^3+1)(q^4+1)$$

The hamming weight w is the number of ones in each code word, but the code words represent the hyperplanes that have the same number of points. Then this give a constant-weight code with hamming weight equals to the number of points in the hyperplane. Since Each hyperplane is a Grassmann geometry of type $A_{3,2}$.

Then by Proposition 2 the hamming weight w is given by:

$$w = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \frac{(q^4-q)(q^4-1)}{(q^2-q)(q^2-1)} = (q^2+q+1)(q^2+1)$$

Let C_1 and C_2 be any two distinct codwords. Since C_1 and C_2 are corresponding to two hyperplanes of $D_{4,2}$ and the hyperplanes are constructions of type $A_{3,2}$ that are considered the symplecta of such geometry, then by Theorem 2 the intersection of any two symplecta is empty or a point. Then any two hyperplanes are disjoint or meeting in a point, this means that $|C_1 \cap C_2| = 0$ or 1. Then the hamming distance between any two codwords either is 2w or 2w-1 (w is the hamming weight of the code), i.e., the minimal distance of the constant code d = 2w-1.

CONCLUSION

This study provided a description of the most important substructures in the point-line geometry of type $D_{4,2}$ such substructures are called hyperplanes. It was found out that there are two types of hyperplanes in this geometry. Those geometrical concepts were used to generate interesting codes called constant-weight codes. The above new results may help to solve some problems related to the two constructions of hyperplanes such as the existence of veldkamp space.

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