Journal of Mathematics and Statistics 3 (4): 220-221, 2007
ISSN 1549-3644

On Characteristic Functions of Fırst Order Theta Function

İsmet Yıldız
University of Bahcesehir Vocational School, Beşiktaş-İstanbul, Turkey

Abstract: In thıs study, a relation on the coefficients periods of first order theta function according to the period pair using the theta characteristic values is established.

Key words: First-theta function, characteristic, period pair

INTRODUCTION

Definition1: For $u \in C$, $\operatorname{Im}\tau \succ 0$ and characteristic $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$, the function defined as

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u,\tau) = \sum_{n=-\infty}^{\infty} \exp\left\{(n+\frac{\varepsilon}{2})^2 \pi i\tau + 2\pi i(n+\frac{\varepsilon}{2})(u+\frac{\varepsilon'}{2}\right\} \qquad (1)$$

is called the first order theta function[1] .

Definition 2: A half-period is half of a period (in particular a complex vector), written

$$\begin{pmatrix}\mu\\ \mu'\end{pmatrix} = \frac{1}{2}\begin{Bmatrix}\mu\\ \mu'\end{Bmatrix} = \frac{\mu'}{2} + \frac{\mu\tau}{2}$$

A reduced half-period is half of a period in which μ and μ' where μ and μ' are integers .

In the present paper, whenever the integers μ and μ' will be as $\mu = 1$ and $\mu' = 1$, unless otherwise stated[2].

In this study,

$$\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix} \equiv \begin{bmatrix}1\\ 1\end{bmatrix},\begin{bmatrix}1\\ 0\end{bmatrix},\begin{bmatrix}0\\ 1\end{bmatrix},\begin{bmatrix}0\\ 0\end{bmatrix} \pmod 2$$

values of characteristic are $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ used. When the periodicity of the function $\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u,\tau)$ for $(1,\tau)$ period pair is examined

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u,\tau) = \sum_{n=-\infty}^{\infty} \exp\left\{(n+\frac{\varepsilon}{2})^2 \pi i\tau + 2\pi i(n+\frac{\varepsilon}{2})(u+1+\frac{\varepsilon'}{2})\right\}$$

$$= \sum_{n=-\infty}^{\infty} \exp\left\{(n+\frac{\varepsilon}{2})^2 \pi i\tau + 2\pi i(n+\frac{\varepsilon}{2})(u+\frac{\varepsilon'}{2}) + n2\pi i + \pi i\varepsilon\right\}$$

$$= (-1)^{\tau}\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u,\tau) = \mu_1\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u,\tau) \quad , for\ \mu_1 = (-1)^{\tau}$$

Also

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u+\tau,\tau) = \sum_{n=-\infty}^{\infty} \exp\left\{(n+\frac{\varepsilon}{2})^2 \pi i\tau + 2\pi i(n+\frac{\varepsilon}{2})(u+\tau+\frac{\varepsilon'}{2})\right\}$$

$$= \sum_{n=-\infty}^{\infty} \exp\left\{(n+\frac{\varepsilon}{2})^2 \pi i\tau + 2\pi i(n+\frac{\varepsilon}{2})(u+\frac{\varepsilon'}{2}) + n2\pi i\tau + \pi i\tau\varepsilon\right\}$$

$$= (-1)^{\tau}\exp\{-\pi i\tau - 2\pi iu\}.\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u,\tau)$$

If we choose $\mu_2 = (-1)^{\tau}\exp\{-\pi i\tau - 2\pi iu\}$ then we obtain the following equality

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u+\tau,\tau) = \mu_2\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u,\tau)$$

Hence

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u+1+\tau,\tau) = \sum_{n=-\infty}^{\infty} \exp\left\{(n+\frac{\varepsilon}{2})^2 \pi i\tau + 2\pi i(n+\frac{\varepsilon}{2})(u+1+\tau+\frac{\varepsilon'}{2})\right\}$$

$$= \sum_{n=-\infty}^{\infty} \exp\left\{(n+\frac{\varepsilon}{2})^2 \pi i\tau + 2\pi i(n+\frac{\varepsilon}{2})(u+\frac{\varepsilon'}{2}) + n2\pi i\tau + \pi i\tau\varepsilon\right\}$$

$$= (-1)^{\tau}\exp\{-\pi i\tau - 2\pi iu - \pi i\varepsilon'\}.\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u,\tau)$$

By using

$$\mu_3 = (-1)^{\tau}\exp\{-\pi i\tau - 2\pi iu - \pi i\varepsilon'\}$$

we obtain

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u+1+\tau,\tau) = \mu_3\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u,\tau)$$

As it is seen here, for $\mu_3 = 1$, because $\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u,\tau)$ is doubly periodic, it would be an elliptic function.

Theorem: For $r \in N^+$

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u+\frac{1}{2^r}+\frac{\tau}{2^r},\tau) = \exp\left\{-\frac{1}{4^r}(\tau+2)\pi i - \frac{1}{2^r}(2u+\varepsilon')\pi i\right\}\theta\begin{bmatrix}\varepsilon+\frac{1}{2^{r-1}}\\ \varepsilon'+\frac{1}{2^{r-1}}\end{bmatrix}(u,\tau)$$ [3]

Proof

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u+\frac{1}{2^r}+\frac{\tau}{2^r},\tau) = \sum_{n=-\infty}^{\infty} \exp\left\{(n+\frac{\varepsilon}{2})^2 \pi i\tau + 2\pi i(n+\frac{\varepsilon}{2})(u+\frac{1}{2^r}+\frac{\tau}{2^r}+\frac{\varepsilon'}{2})\right\}$$

$$= \sum_{n=-\infty}^{\infty} \exp\left\{(n+\frac{\varepsilon}{2})^2 \pi i\tau + 2\pi i(n+\frac{\varepsilon}{2})(u+\frac{\varepsilon'}{2}) + (\frac{n\pi i\tau}{2^{r-1}} + \frac{n\pi i}{2^{r-1}} + \frac{\pi i\tau\varepsilon}{2^r} + \frac{\pi i\varepsilon}{2^r})\right\} \qquad (2)$$

Corresponding Author: İsmet Yıldız, University of Bahcesehir Vocational School, Beşiktaş-İstanbul, Turkey

On the other hand

$$\theta\begin{bmatrix}\varepsilon+\frac{1}{2^{r-1}}\\ \varepsilon'+\frac{1}{2^{r-1}}\end{bmatrix}(u,\tau)$$

$$=\sum_{n=-\infty}^{\infty}\exp\left\{\begin{matrix}(n+\frac{\varepsilon}{2})^2\pi i\tau+2\pi i(n+\frac{\varepsilon}{2})(u+\frac{\varepsilon'}{2})+\frac{2\pi i\tau}{2^{r-1}}+\frac{2\pi i\varepsilon'}{2^{r-1}}+\frac{n\pi i\tau}{2^{r-1}}+\frac{n\pi i\tau}{2^{r-1}}\\ +\frac{\pi i\tau\varepsilon}{2^r}+\frac{n\pi i}{2^{r-1}}+\frac{\pi i\tau}{4^r}+\frac{2\pi i}{4^r}+\frac{\pi i\varepsilon}{2^r}\end{matrix}\right\}$$

and

$$\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i-\frac{1}{2^r}(2u+\varepsilon')\pi i\right\}\theta\begin{bmatrix}\varepsilon+\frac{1}{2^{r-1}}\\ \varepsilon'+\frac{1}{2^{r-1}}\end{bmatrix}(u,\tau)$$

$$=\sum_{n=-\infty}^{\infty}\exp\left\{(n+\frac{\varepsilon}{2})^2\pi i\tau+2\pi i(n+\frac{\varepsilon}{2})(u+\frac{\varepsilon'}{2})+\frac{n\pi i\tau}{2^{r-1}}+\frac{n\pi i}{2^{r-1}}+\frac{\pi i\tau\varepsilon}{2^r}+\frac{\pi i\varepsilon}{2^r})\right\} \quad (3)$$

Using the equalities of (2) and (3) for

$$\mu_4=\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i-\frac{1}{2^r}(2u+\varepsilon')\pi i\right\}$$

we obtain the following equality

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u+\frac{1}{2^r}+\frac{\tau}{2^r},\tau)=\mu_4\theta\begin{bmatrix}\varepsilon+\frac{1}{2^{r-1}}\\ \varepsilon'+\frac{1}{2^{r-1}}\end{bmatrix}(u,\tau)$$

By the theorem given above we can obtain the following characteristic equalities for u = 0 value of the complex variable

(a)
$$\theta\begin{bmatrix}1\\1\end{bmatrix}(0+\frac{1}{2^r}+\frac{\tau}{2^r},\tau)=\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i-\frac{1}{2^r}\pi i\right\}\theta\begin{bmatrix}1+\frac{1}{2^{r-1}}\\ 1+\frac{1}{2^{r-1}}\end{bmatrix}(0,\tau)$$

$$=\sum_{n=-\infty}^{\infty}\exp\left\{(n+\frac{1}{2}+\frac{1}{2^r})^2\pi i\tau+2\pi i(n+\frac{1}{2}+\frac{1}{2^r})(0+\frac{1}{2}+\frac{1}{2^r})-\frac{\pi i\tau}{4^r}-\frac{\pi i}{2^{2r-1}}-\frac{\pi i}{2^r}\right\}$$

$$=\sum_{n=-\infty}^{\infty}\exp\left\{(n+\frac{1}{2})^2\pi i\tau+\frac{n\pi i\tau}{2^{r-1}}+\frac{\pi i\tau}{2^r}+\frac{n\pi i}{2^{r-1}}+\frac{\pi i}{2^r}+\frac{\pi i}{2}+n\pi i\right\} \quad (4)$$

$$\theta\begin{bmatrix}1\\1\end{bmatrix}(0+\frac{1}{2^r}+\frac{\tau}{2^r},\tau)=\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i-\frac{1}{2^r}\pi i\right\}\theta\begin{bmatrix}1+\frac{1}{2^{r-1}}\\ 0+\frac{1}{2^{r-1}}\end{bmatrix}(0,\tau)$$

$$=\sum_{n=-\infty}^{\infty}\exp\left\{(n+\frac{1}{2}+\frac{1}{2^r})^2\pi i\tau+2\pi i(n+\frac{1}{2}+\frac{1}{2^r})(0+\frac{1}{2}+\frac{1}{2^r})-\frac{\pi i\tau}{4^r}-\frac{\pi i}{2^{2r-1}}\right\}$$

$$=\sum_{n=-\infty}^{\infty}\exp\left\{(n+\frac{1}{2})^2\pi i\tau+\frac{n\pi i\tau}{2^{r-1}}+\frac{\pi i\tau}{2^r}+\frac{n\pi i}{2^{r-1}}+\frac{\pi i}{2^r}+\frac{\pi i}{2}+n\pi i\right\} \quad (5)$$

From the equation (4) and (5), we can get the following equality

$$\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i-\frac{1}{2^r}\pi i\right\}\theta\begin{bmatrix}1+\frac{1}{2^{r-1}}\\ 1+\frac{1}{2^{r-1}}\end{bmatrix}(0,\tau)=\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i\right\}\theta\begin{bmatrix}1+\frac{1}{2^{r-1}}\\ 0+\frac{1}{2^{r-1}}\end{bmatrix}(0,\tau)$$

(b)
$$\theta\begin{bmatrix}0\\0\end{bmatrix}(0+\frac{1}{2^r}+\frac{\tau}{2^r},\tau)=\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i\right\}\theta\begin{bmatrix}0+\frac{1}{2^{r-1}}\\ 0+\frac{1}{2^{r-1}}\end{bmatrix}(0,\tau)$$

$$=\sum_{n=-\infty}^{\infty}\exp\left\{(n+\frac{1}{2^r})^2\pi i\tau+2\pi i(n+\frac{1}{2^r})(0+\frac{1}{2^r})-\frac{\pi i\tau}{4^r}-\frac{\pi i}{2^{2r-1}}\right\}$$

$$=\sum_{n=-\infty}^{\infty}\exp\left\{n^2\pi i\tau+\frac{n\pi i\tau}{2^{r-1}}+\frac{n\pi i}{2^{r-1}}\right\}$$

$$\theta\begin{bmatrix}0\\1\end{bmatrix}(0+\frac{1}{2^r}+\frac{\tau}{2^r},\tau)=\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i-\frac{\pi i}{2^r}\right\}\theta\begin{bmatrix}0+\frac{1}{2^{r-1}}\\ 1+\frac{1}{2^{r-1}}\end{bmatrix}(0,\tau)$$

$$=\sum_{n=-\infty}^{\infty}\exp\left\{(n+\frac{1}{2^r})^2\pi i\tau+2\pi i(n+\frac{1}{2^r})(0+\frac{1}{2}+\frac{1}{2^r})-\frac{\pi i\tau}{4^r}-\frac{\pi i}{2^{2r-1}}-\frac{\pi i}{2^r}\right\}$$

$$=\sum_{n=-\infty}^{\infty}\exp\left\{n^2\pi i\tau+\frac{n\pi i\tau}{2^{r-1}}+n\pi i+\frac{n\pi i}{2^{r-1}}\right\} \quad (7)$$

If $n=2k\in N^+$, then form the equalities (6) and (7) the following is obtained

$$\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i-\frac{\pi i}{2^r}\right\}\theta\begin{bmatrix}0+\frac{1}{2^{r-1}}\\ 1+\frac{1}{2^{r-1}}\end{bmatrix}(0,\tau)=\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i\right\}\theta\begin{bmatrix}0+\frac{1}{2^{r-1}}\\ 0+\frac{1}{2^{r-1}}\end{bmatrix}(0,\tau)$$

CONCLUSION

With the help of this theorem proved above, transformations among theta functions can be found for characteristic values $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ according to all multiples $\frac{1}{2^r}$ of the periods.

For example; if r = 4, then it follows that

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u+\frac{1}{16}+\frac{\tau}{16},\tau)=\exp\left\{-\frac{1}{256}(\tau+2)\pi i-\frac{1}{16}(2u+\varepsilon')\pi i\right\}\theta\begin{bmatrix}\varepsilon+\frac{1}{8}\\ \varepsilon'+\frac{1}{8}\end{bmatrix}(u,\tau)$$

The subject that should be discussed here is; characteristic values $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\equiv\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}(\text{mod}\,2)$ of first order theta function can be expressed as characteristic values $\theta\begin{bmatrix}\varepsilon+\frac{1}{2^{r-1}}\\ \varepsilon'+\frac{1}{2^{r-1}}\end{bmatrix}$.

REFERENCES

1. Harry, E.R., 1973. Elliptic functions, Theta Functions and Riemann Surfaces. The Williams and Wilkins Company Baltimore, Maryland, pp: 75-79.
2. Yıldız, İ., 2004. On extension of the modular transformations over the modular group by reflection. Appl. Math. Comput., 153: 111-116.
3. Yıldız, İ., 2005. On the elliptic function arising from the theta functions and dedekind function. J. Math. & Stat., 1: 153 –159.