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A General Probability Distribution Using Bürmann Power Series

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Abstract: The goal of this study is to present a general power series distribution that exhibits the properties of some well known distributions. To accomplish this goal we examine an infinite sequence of independent random variables having a Bürmann's power series distribution. Consequently, we derive moment generating function of the distribution and establish the maximum likelihood estimate of the component that can be attributed to the parameter of the distribution. Using the results mentioned above we verify our conjecture on two known parametric discrete distributions, the Poisson and the Binomial.

Key words: Binomial distribution, moment generating function, maximum likelihood estimators, poisson distribution

INTRODUCTION

In 1927 Whittaker and Watson presented a definition for the Bürmann series in their study of analytic function theory as a general power series. The formal definition and basic results are given in[1]. It should be noted that in the definition of the series there is a general function sequence $h_n(x)$ which can be uniformly estimated to be x. This reduces to the traditional methods using power series of x[2]. Researchers such as Awad and Alawneh[3], Kyriakoussis and Vamvakari[4] and Shanmugam[5] developed continuous and discrete distributions based on power series of x. However none has developed methods using the Bürmann series. Other researchers such as King[6] and Patterson, *et. al.*[7] developed methods of summability theory using this series. The strength of the Bürmann series lies in the flexibility of selecting $h_n(x)$. In this article we derive moment generating functions and maximum likelihood estimators for parameter of distributions based on the Bürmann series. We will illustrate our result using Binomial and Poisson distributions.

Definition 1: A Bürmann-series is a series of this form:

$$f_n(x) = \sum_{k=0}^{\infty} b_k (h_n(x))^k$$

where f and h are given functions.

Additional results can also be found in[8]. This notion was extended to Probability Theory by J. P. King in[6] by considering the following definition.

Definition 2: A random variable X_n with range $\{0,1,2,...\}$ has a Bürmann-series distribution if there exists a Bürmann-series

$$s_n(x) = \sum_{k=0}^{\infty} b_{n,k} (h_n(x))^k$$

which is convergent on some set I with $b_{n,k} \geq 0, s_n(x) \neq 0$ for $x \in I$ and

$$P(X_n = k) = \frac{b_{n,k}(h_n(x))^k}{s_n(x)}, k = 0,1,2,....$$

RESULTS

Section 2.1: Moment generating function is one of the most useful instrument use to derive moments for a probability density function. We use the implicit definition of the probability function for the Bürmann's power series to derive its moment generating function. The derivation requires expansion of the infinite series and regrouping of the terms as shown in the proof of the following theorem.

Theorem 1: Let $\{X_n\}$ be an infinite sequence of random variables, each having range $\{0,1,2,...\}$ and with the following Bürmann-series distribution

$$P(X_n = k) = \frac{b_{n,k}(h_n(x))^k}{s_n(x)}, k = 0,1,2,... \qquad (1)$$

where

$$s_n(x) = \sum_{k=0}^{\infty} b_{n,k} (h_n(x))^k, n = 0,1,2,3,....$$

Suppose each $h_n(x)$, $n = 0,1,2,3,...$ is infinitely

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differentiable, then the moment generating function of X_n has the subsequent form

$$M_k(t) = \frac{U(h_n(x)e^t)}{U(h_n(x))}, \quad (2)$$

where $U(x) = \sum_{k=0}^{\infty} b_{n,k} x^k$. The p-th moment is generated by the following expression

$$M_k^p(0) = \frac{\sum_{k=0}^{\infty} b_{n,k} k^p (h_n(x))^k}{s_n(x)}.$$

Proof: Let $\{X_n\}$ be defined as in Definition 1.2 and consider $E(e^{tk})$, which is denoted by $M_k(t)$. Note that $M_k(t)$ has the following expansion.

$$M_k(t) = E(e^{tk})$$

$$= \frac{1}{s_n(x)} \sum_{k=0}^{\infty} b_{n,k} e^{tk} (h_n(x))^k$$

$$= \frac{1}{s_n(x)} \sum_{k=0}^{\infty} b_{n,k} \left\{ \begin{matrix} 1 + tk + \frac{(tk)^2}{2!} \\ + \frac{(tk)^3}{3!} + \cdots \end{matrix} \right\} (h_n(x))^k$$

$$= \frac{1}{s_n(x)} \begin{bmatrix} b_{n,0} + b_{n,1} e^t h_n(x) \\ + b_{n,2} e^{2t} h_n^2(x) + b_{n,3} e^{3t} h_n^3(x) + \cdots \end{bmatrix}$$

$$= \frac{1}{s_n(x)} \Big[b_{n,0} + 0 + \cdots$$

$$+ b_{n,1} h_n(x) + b_{n,1} h_n(x) t + b_{n,1} h_n(x) \frac{t^2}{2!}$$

$$+ b_{n,1} h_n(x) \frac{t^3}{3!} + \cdots$$

$$+ b_{n,p} h_n^p(x) + b_{n,p} h_n^p(x) pt$$

$$+ b_{n,p} h_n^p(x) \frac{(pt)^2}{2!} + b_{n,p} h_n^p(x) \frac{(pt)^3}{3!} + \cdots \Big]$$

$$= \frac{1}{s_n(x)} \begin{bmatrix} \sum_{k=0}^{\infty} b_{n,k} \frac{1}{0!} + t \sum_{k=0}^{\infty} b_{n,k} \frac{k}{1!} \\ + t^2 \sum_{k=0}^{\infty} b_{n,k} \frac{k^2}{2!} \end{bmatrix}$$

$$+ \cdots + t^p \sum_{k=0}^{\infty} b_{n,k} \frac{k^p}{p!} \cdots \Big]$$

$$= \frac{1}{s_n(x)} \sum_{i=0}^{\infty} \frac{t^i}{i!} \left(\sum_{k=0}^{\infty} b_{n,k} k^i (h_n(x))^k \right)$$

$$= \sum_{i=0}^{\infty} \frac{t^i}{i!} \frac{\sum_{k=0}^{\infty} b_{n,k} k^i (h_n(x))^k}{s_n(x)}.$$

$$= \frac{1}{s_n(x)} \begin{bmatrix} b_{n,0} + b_{n,1} e^t h_n(x) \\ + b_{n,2} e^{2t} h_n^2(x) + b_{n,3} e^{3t} h_n^3(x) + \cdots \end{bmatrix}$$

$$= \frac{1}{s_n(x)} \Big[b_{n,0} + 0 + \cdots$$

$$+ b_{n,1} h_n(x) + b_{n,1} h_n(x) t + b_{n,1} h_n(x) \frac{t^2}{2!}$$

$$+ b_{n,1} h_n(x) \frac{t^3}{3!} + \cdots$$

$$+ b_{n,2} h_n^2(x) + b_{n,2} h_n^2(x) 2t + b_{n,2} h_n^2(x) \frac{(2t)^2}{2!}$$

$$+ b_{n,2} h_n^2(x) \frac{(2t)^3}{3!} + \cdots$$

$$\vdots$$

$$+ b_{n,p} h_n^p(x) + b_{n,p} h_n^p(x) pt + b_{n,p} h_n^p(x) \frac{(pt)^2}{2!}$$

$$+ b_{n,p} h_n^p(x) \frac{(pt)^3}{3!} + \cdots$$

$$\vdots \; \Big]$$

$$= \frac{1}{s_n(x)} \left[\sum_{k=0}^{\infty} b_{n,k} \frac{1}{0!} + t \sum_{k=0}^{\infty} b_{n,k} \frac{k}{1!} + t^2 \sum_{k=0}^{\infty} b_{n,k} \frac{k^2}{2!} \right.$$

$$\left. + \cdots + t^p \sum_{k=0}^{\infty} b_{n,k} \frac{k^p}{p!} \cdots \right]$$

$$= \frac{1}{s_n(x)} \sum_{i=0}^{\infty} \frac{t^i}{i!} \left(\sum_{k=0}^{\infty} b_{n,k} k^i (h_n(x))^k \right)$$

$$= \sum_{i=0}^{\infty} \frac{t^i}{i!} \frac{\sum_{k=0}^{\infty} b_{n,k} k^i (h_n(x))^k}{s_n(x)}.$$

Now consider the p-th derivative of Equation 2,

$$\frac{d^p}{dt^p} M_k(t) = \frac{1}{U(h_n(x))} \frac{d^p}{dt^p} U(e^t h_n(x)).$$

Using the definition of U we obtain the expression for the p-th derivative as

$$\frac{d^p}{dt^p} M_k(t) = \frac{1}{U(h_n(x))} U(k^{\frac{p}{k}} e^t h_n(x)).$$

Consequently, the p-th moment of the probability function in Equation 2.1 is

$$E(X_n^p) = \frac{1}{U(h_n(x))},$$
$$U(k^{\frac{p}{k}} h_n(x)) = \frac{\sum_{k=0}^{\infty} b_{n,k} k^p (h_n(x))^k}{s_n(x)}. \quad (3)$$

This completes the proof of the theorem.

We use the results in Theorem 2.1 to illustrate the moment generating functions of two special cases for Bürmann-series distribution as in[6].

Example 1: Given the conditions in Theorem 2.1 with

$$U(x) = \sum_{k=0}^{\infty} b_{n,k} x_n^k$$
$$b_{n,k} = \begin{cases} \binom{n}{k} & \text{if} \quad k \le n, \\ 0 & \text{if} \quad \text{otherwise.} \end{cases}$$

We are granted the following moment generating function

$$M_k(t) = \frac{(1+e^t h(x))^n}{s_n(x)},$$

and if $h_n(x) = \frac{x}{1-x}$, then $s_n(x) = \frac{1}{(1-x)^n}$. This yields the Binomial moment generating function.

Example 2: Given the conditions in Theorem 2.1 with $U(x) = \sum_{k=0}^{\infty} b_{n,k} x_n^k$ and

$b_{n,k} = \frac{1}{k!}$.

We are granted the following moment generating function

$$M_k(t) = \frac{1}{s_n(x)} e^{e^t h_n(x)}.$$

If $h_n(x) = nx$ then $s_n(x) = e^{nx}$. This yields the Poisson moment generating function.

Section 2.2: Using the moment generating function in Sections 2.1 we prove the following results of regular moments for $\{X_n\}$, which is similar to the results established by King in[6].

Theorem 2: Let *{Xn}* be defined as in Theorem 2.1 and $h_n'(x) \neq 0$. Let E and V denote, respectively, the expectation operator and the variance operator. Also let $\mu_n = E(X_n)$. Then, for each *n=0, 1, 2, 3,...* and each *m=1, 2, 3, ...*

$$E(X_n) = \frac{h_n(x)}{h_n'(x)} \cdot \frac{s_n'(x)}{s_n(x)},$$
$$E(X_n^{m+1}) = \frac{h_n(x)}{h_n'(x)} \frac{d}{dx}(E(X_n^m))$$
$$+E(X_n)E(X_n^m),$$

and

$$V(X_n) = \frac{h_n(x)}{h_n'(x)} \frac{d}{dx} E(X_n).$$

Proof: Equation 3 yields the following,

$$\begin{aligned} E(X_n) &= \frac{U(k^{\frac{1}{k}} h_n(x))}{U(h_n(x))} \\ &= \frac{1}{s_n(x)} \sum_{k=1}^{\infty} k b_{n,k} (h_n(x))^k \\ &= \frac{h_n(x)}{h_n'(x)} \frac{1}{s_n(x)} \sum_{k=1}^{\infty} k b_{n,k} (h_n(x))^{k-1} h_n'(x) \\ &= \frac{h_n(x)}{h_n'(x)} \frac{s_n'(x)}{s_n(x)}. \end{aligned}$$

Thus part (1) of this theorem is established. Using Equation 3 again we obtain the m -th moment as

$$\begin{aligned} E(X_n^{m+1}) &= \frac{U(k^{\frac{m+1}{k}} h_n(x))}{U(h_n(x))} \\ &= \frac{1}{s_n(x)} \sum_{k=1}^{\infty} k^{m+1} b_{n,k} (h_n(x))^k \\ &= \frac{h_n(x)}{h_n'(x)} \frac{1}{s_n(x)} \sum_{k=1}^{\infty} k^{m+1} b_{n,k} (h_n(x))^{k-1} h_n'(x) \\ &= \frac{h_n(x)}{h_n'(x)} \Bigg[\frac{1}{s_n(x)} \sum_{k=1}^{\infty} k^{m+1} b_{n,k} (h_n(x))^{k-1} h_n'(x) \\ &\quad - \frac{s_n(x)}{s_n'(x)} \frac{1}{s_n(x)} \sum_{k=1}^{\infty} k^m b_{n,k} (h_n(x))^k \\ &\quad + \frac{s_n(x)}{s_n'(x)} \frac{1}{s_n(x)} \sum_{k=1}^{\infty} k^m b_{n,k} (h_n(x))^k \Bigg] \\ &= \frac{h_n(x)}{h_n'(x)} \frac{d}{dx} M_k^m(0) + M_k(0) M_k^m(0). \end{aligned}$$

Thus part (2) of the theorem is completed. Note that part (3) is a special case of part (2) when $m = 1$. The assumption $h_n'(x) \neq 0$, allows the moments derived above to be expressed in closed forms, however the assumption is not necessary.

Section 2.3: The maximum likelihood is one of the most important characteristics of an estimator for a given probability distribution function. We shall use this probability distribution to establish the maximum likelihood estimators for the parameters in the underlying probability function for the Bürmann probability distribution. Let $L(k,h_n(x))$ denote the likelihood function for Equation 2. The maximum likelihood estimate of $h_n(x)$ can be derived as shown in the following theorem.

Theorem 3: Let $X_1 = k_1, X_2 = k_2, \ldots, X_n = k_n$ be a sequence of random variables defined for the Bürmann-series distribution function then the maximum likelihood estimator of $h_n(x)$ is given by

$$\hat{h}_n(x) = \frac{h_n^{'}(x)s_n(x)}{s_n^{'}(x)}\sum_{i=1}^{n} k_i. \tag{4}$$

Proof: For Equation 2 the likelihood function is given by

$$L(k,h_n(x)) = \prod_{k=1}^{n} \frac{b_{n,k}h_n^{k_i}(x)}{s_n(x)}.$$

After taking the first derivative we obtain

$$\frac{d}{dx}\ln L(k,h_n(x)) = -\frac{s_n^{'}(x)}{s_n(x)} + \frac{h_n^{'}(x)}{h_n(x)}\sum_{i=1}^{n} k_i.$$

Thus equating the right hand side of the expression to zero, we get Equation 2.4. This completes the proof of Theorem 2.3.

We apply the result obtained here to derive the maximum likelihood estimators for parameters of the same two special cases described in Section 2.1.

Example 3: In the first case let, $h_n(x) = nx$ and $s_n(x) = e^{nx}$, thus Equation 4 gives

$$\hat{h}_n(x) = \sum_{i=1}^{n} k_i,$$

which is a sufficient statistic for the parameter involved in the Poisson distribution[9].

Example 4: In the second case let, $h_n(x) = \frac{x}{1-x}$ and $s_n(x) = \frac{1}{(1-x)^n}$, thus Equation 4 gives

$$\hat{h}_n(x) = \frac{\sum_{i=1}^{n} k_i}{n - \sum_{i=1}^{n} k_i},$$

which is a sufficient statistic for the parameter involved in the Binomial distribution[9] and is consistent with the definition of $h_n(x)$.

CONCLUSION

In this study we have established some useful results on Bürmann power series distribution. The discrete nature on the probability function of the series enables us to illustrate the results for the known parametric discrete distributions. We are able to impose conditions on $b_{n,k}$ and $h_n(x)$ to generate the moments and estimates of the parameters for the Poisson and the Binomial functions.

The results in this study add new ideas to the underlying theory of probability. This was accomplished by presenting moment generating function and maximum likelihood estimators of the power series. There are many possible directions for these ideas to venture, not the least of which is to consider its connection with testing of hypotheses. The analysis here is based on Bürmann power series distribution, but we can consider any other general power series.

The advantage of this generalized distribution lies in its flexibility of selection of $h_n(x)$. Because of this property the general probability distribution can be viewed as continuous or discrete. The method presented here is classical in nature. Feller presented in[2] one general class of power series distribution and here we presented a class that includes his results.

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