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Koch Curves: Rewriting System, Geometry and Application

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Abstract: Problem statement: Recently, new Koch curves have been generated by dividing the initiator into three unequal parts. There is no formal rewriting system to generate such kind of curves. **Approach:** It is required to measure the new changed geometrical properties. Generalized rewriting systems for the new Koch curves have been developed. **Results:** New formulas have been given to measure their geometrical properties. **Conclusion/Recommendations:** The geometrical properties of new Koch curves make them more suitable as antennas in wireless communication than the conventional Koch curve.

Key words: Koch curve, superior koch curve, koch loop, rewriting system, fractal antenna, fractal dimension, rewriting rules, fundamental mathematical properties, natural number, superior fractals, multi-band characteristics

INTRODUCTION

The classical Koch curve is, mathematically, continuous everywhere but differentiable nowhere. It is an example of bounded curve of infinite length (Schroeder, 2009; Falconer, 2003). Koch snowflake curve is used as a fractal antenna. Nathan (2005) described the importance of fractal antennas explicitly for wireless technologies with emphasis to military services. For a detailed study on Koch fractal antenna and related various new findings, one may refer to (Elkamchouchi and Nasr, 2007; Ghatak *et al*., 2009; Kordzadeh and Kashani, 2009; Krishna *et al*., 2009; Mirzapour and Hassani, 2009; Song *et al*., 2008; Werner and Suman, 2003; Zhang and Kishk, 2006) and several cross references thereof. McClure (2008) has discussed the vibration modes of a drum shaped like a Koch snowflake. Epstien and Adeeb (2008) derived stiffness of the Koch curve. Jibrael *et al*. (2008) simulated quadratic Koch antenna and explored its antenna properties. Further, Vinoy *et al*. (2002) presented a new way of generation of variants of a Koch curve by varying indentation angle and gave a formula to calculate their fractal dimension and studied the impact of fractal dimension in the design of multi-resonant fractal antennas (Vinoy *et al*., 2004). Generated new Koch curves as new examples of superior fractals by dividing the initiator into three unequal parts. A comprehensive review of literature on superior fractals, which are constructed using superior iterates, is given by Singh *et al*. (2011).

The purpose of this study is to develop rewriting rules for superior Koch curves and suggest formulas for calculation of their fundamental mathematical properties.

Preliminaries: Construction of the Koch curve (Fig. 1) is well known. It can be expressed by following rewrite system (L-system) (http://en.wikipedia.org/wiki/Koch_curve):

$$\begin{aligned}&\text{Alphabet}: F\\&\text{Constant}: +, -\\&\text{Axiom}: F\\&\text{Production rule}: F \rightarrow F+F--F+F\end{aligned} \quad (1)$$

Here, F means "draw forward", + means "turn left 60°" and - means "turn right 60°".
To draw a Koch snowflake curve, the Prod. Rule 1 is applied on axiom “F - - F - - F”.

Rewriting system for koch curves: In this section, we develop general production rules to draw superior Koch curves. We divide the production rules for Koch curves at different scaling factors into two parts. All the symbols used in the following rewriting systems carry similar meanings as in that of Koch

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curve. We use the same initiator, as shown in Fig. 1a, in all the generations.

Case 1: Koch curves at scaling factor $S = \frac{1}{n}$, where n is an odd number.

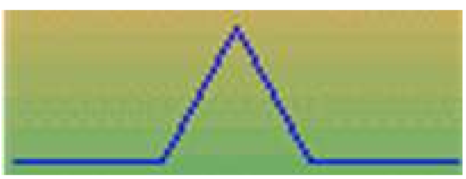
Let $S = \frac{1}{5}$, then one of the possible Koch curves is shown in Fig. 2. The following is the rewriting system for such a Koch curve:

Figure 2 Koch middle one-fifth curve for (r1, r2, r3) = (2/5, 1/5, 2/5) with its initiator:

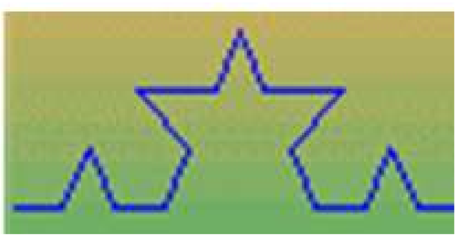
$$\begin{aligned} &\text{Scaling factor : Alphabet : F} \\ &\text{Constant : } +, - \\ &\text{Axiom : F} \\ &\text{Production rule : } F \rightarrow FF + F - - F + FF \end{aligned} \quad (2)$$

At $S = \frac{1}{7}$, rewriting system for Koch middle one-seventh curve is as follows:

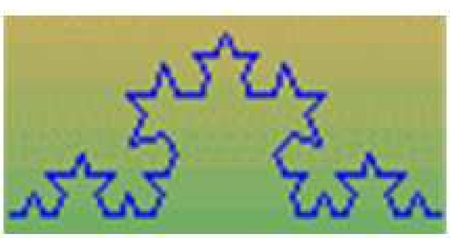
$$\begin{aligned} &\text{Scaling factor : Alphabet : F} \\ &\text{Constant : } +, - \\ &\text{Axiom : F} \\ &\text{Production rule : } F \rightarrow FFF + F - - F + FFF \end{aligned} \quad (3)$$



(a)



(b)



(c)

Fig. 1: Iterative construction of koch curve

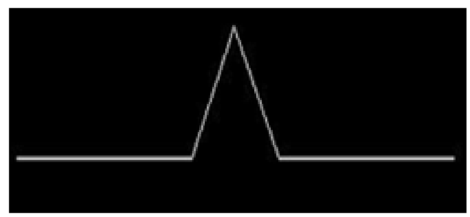
The general rewriting system for Koch curves at S=1/n, where n = 3, 5, 7, 9, …, which can be derived from Prod. Rule 1, 2 and 3 is as follows:

Scaling factor $S = \frac{1}{n}$, where n is an odd natural number and $n \geq 3$:

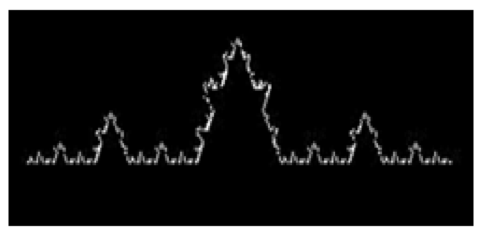
$$\begin{aligned} &\text{Alphabet : F} \\ &\text{Constant : } +, - \\ &\text{Axiom : F} \\ &\text{Production rule : } F \rightarrow ((n-1)/2)F + F - - F + ((n-1)/2)F \end{aligned} \quad (4)$$

Case 2: Koch curves at scaling factor S=1/n, where n is an even number.

Let the scaling factor be ¼, then rewriting system for the two possible Koch middle one-fourth curves (see the generators in Fig. 3) is given below.



(a)



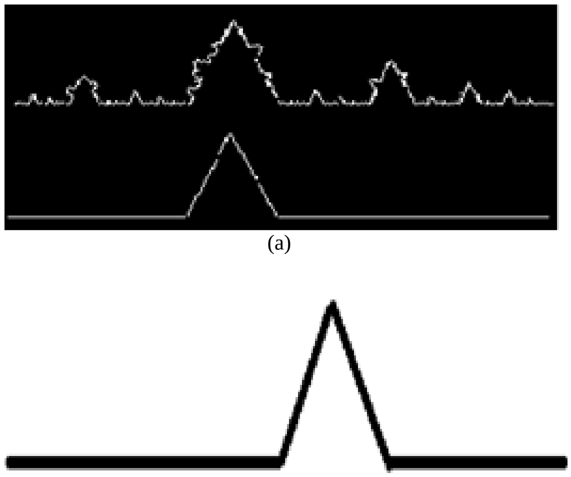
(b)

Fig. 2: Koch middle one-fifth curve for (r1, r2, r3) = (2/5, 1/5, 2/5) with its initiator

(a)

(b)

Fig. 3: Generators of two possible Koch middle one-fourth curves (a) Generator at (r1, r2, r3) = (1/4, 1/4, 1/2) (b) Generator at (r1, r2, r3) = (1/2, 1/4, 1/4)



(a)

(b)

Fig. 4: Koch middle one-sixth curve (a) At generator (r1, r2, r3) = (1/3, 1/6, 1/2) (b)At generator (r1, r2, r3) = (1/2, 1/6, 1/3)

Scaling factor: $S = \frac{1}{4}$ Alphabet: F:

$$\begin{aligned}&\text{Constant}: +, -\\&\text{Axiom}: F\\&\text{Production rule}: F \rightarrow F + F - - F + FF\\&\text{or } F \rightarrow FF + F - - F + F\end{aligned} \quad (5)$$

At the scaling factor $S = \frac{1}{6}$, rewriting system for two possible Koch middle one-sixth curve (Fig. 4) is as follows:

$$\begin{aligned}&\text{Scaling factor}:\\&\text{Alphabet}: F\\&\text{Constant}: +, -\\&\text{Axiom}: F\\&\text{Production rule}: F \rightarrow FF + F - - F + FFF\\&\text{or } F \rightarrow FFF + F - - F + FF\end{aligned} \quad (6)$$

The general rewriting system for Koch curves at $S = \frac{1}{n}$, where n = 4, 6, 8, 10, …, which can be derived from Prod. Rule 5 and 6 is as follows:

Scaling factor: $S = \frac{1}{n}$, where n is an even natural number and $n \geq 4$:

$$\begin{aligned}&\text{Alphabet}: F\\&\text{Constant}: +, -\\&\text{Axiom}: F\\&\text{Production rule}: F \rightarrow (n/2 - 1)F + F - - F + ((n/2)F\\&\text{Or } F \rightarrow (n/2 - 1)F + F - - F + ((n/2)F\end{aligned} \quad (7)$$

Applications to fractal antenna: Authors have generated superior Koch curves at scaling factors s = 1/n, for n≥4. Koch loop can be generated by applying Prod. Rule 4 or 7 on the axiom "F - - F - - F". Following are some of the geometric features of a Koch loop. Here, r denotes the radius of the circle which accommodates the Koch loop.

The general formula to calculate the area of a Koch loop is given by:

$$\text{Area}_{\text{KochLoop}} = \frac{3\sqrt{3}}{4} r^2 (1 + \frac{3}{n^2 - 4}), n \in N \text{ and } n \geq 3 \quad (8)$$

Therefore:

$$\text{Area}_{\text{KochLoop}} = \lim_{n \to \infty} \frac{3\sqrt{3}}{4} r^2 (1 + \frac{3}{n^2 - 4}) = \frac{3\sqrt{3}}{4} r^2$$

Thus smaller the scaling factor, lesser the area of the Koch loops.

The general formula to calculate the perimeter of a Koch loop is given by:

$$\text{Perimeter}_{\text{KochLoop}} = 3\sqrt{3}r + \frac{9\sqrt{3}}{n} r ((4/3)^m - 1), m, n \in N \text{ and } n \geq 3 \quad (9)$$

Where m is the total number of iterations.

Considering:

$$\lim_{m \to \infty} 3\sqrt{3}r \frac{9\sqrt{3}}{n} r((4/3)^m - 1)$$

We see that $\text{Perimeter}_{\text{KochLoop}} = \infty$

Theoretically, a Koch loop can accommodate wire of infinite length. However, with the reducing scaling factor, growth rate of the perimeter decreases.

The general formula to calculate the dimension of a Koch curve is given by:

$$\text{Dimension}_{\text{Kochcurve}} = \frac{\log(n+1)}{\log n}, n \in N \text{ and } n \geq 3 \quad (10)$$

From (10), it can be calculated that the dimension of Koch curves at n = 4, 5 and 6 is approximately 1.16, 1.113 and 1.086 respectively. According to (Vinoy *et al*., 2004; Song *et al*., 2008), since fractal antennas having smaller dimensions show better multi-band characteristics, Koch antennas at $S = \frac{1}{n}$, for n ≥ 4, shall exhibit better multiband characteristics.

Theoretically, superior Koch loops, generated at $S = \frac{1}{n}$, for n ≥ 4, are more suitable as antennas than the Koch antenna at $S = \frac{1}{3}$, as they are more compact in size (cf. (8)) and can accommodate long length of wire (cf. (9)) and therefore will have better multiband characteristics due to smaller fractal dimension (cf. (10)).

CONCLUSION

Following formulas have been derived in this study:

Rewriting system for superior Koch curves at.

Scaling factor: s = 1/n, where n is an odd natural number and n ≥ 3:

$$\begin{aligned}&\text{Alphabet}: F\\&\text{Constant}: +, -\\&\text{Axiom}: F\\&\text{Production rule}: F \rightarrow ((n-1)/2F + F - -F + ((n-1)/2)F\end{aligned}$$

Scaling factor: s= 1/n, where n is an even natural number and n ≥ 4:

$$\begin{aligned}&\text{Alphabet}: F\\&\text{Constant}: +, -\\&\text{Axiom}: F\\&\text{Production rule}: F \rightarrow (n/2-1)F + F - -F + (n/2)F\\&\text{or } F \rightarrow (n/2)F + F - -F + (n/2-1)F\end{aligned}$$

Geometrical properties of a superior Koch loop is given by:

$$\text{Area}_{\text{KochLoop}} = \frac{3\sqrt{3}}{4} r^2 \left(1 + \frac{3}{n^2-4}\right), \forall n \in N \text{ and } n \geq 3;)$$

$$\text{Permiter}_{\text{KochLoop}} = 3\sqrt{3}r + \frac{9\sqrt{3}}{n} r((4/3)^m - 1),$$
$$\forall m, n \in N \text{ and } n \geq 3;$$

$$\text{Dimension}_{\text{KochLoop}} = \frac{\log(n+1)}{\log n}, \forall n \in N, n \geq 3;$$

Here, $\frac{1}{n}$ is the scaling factor and r is the radius of the circle which accommodates the Koch loop.

Fractal antenna is preferred over circular antenna because it is compact and has multiband characteristics. Theoretically, a superior Koch antenna, generated at $S = \frac{1}{n}$, n ≥ 4, are more suitable as an antenna than the Koch antenna at $S = \frac{1}{3}$, as they are more compact (Result 2(i)), can accommodate long length of wire (Result 2(ii)) and will have better multiband characteristic due to smaller fractal dimension (Result 2(iii)).

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