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On the Stability of the Triangular Lagrangian Equilibrium Points in the Relativistic Restricted Three–Body Problem

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Abstract: The restricted three body problem is treated in the framework of the post-Newtonian approximation of general relativity. The equations of motion are linear zed around the libration points $L_{4,5}$. Locations of the equilibrium points $L_{4,5}$ are obtained. Existence and stability of these points are investigated. The relativistic correction to the classical mass ratio is determined

Key words: Relativistic RTBP, Libration points, Stability

INTRODUCTION

In any assumed isolated two-body massive orbiting system (such as the Sun and the Earth), there are five equilibrium points, $L_i, i = 1,2,3,4,5,$ these points usually called Lagrangian or Libration points. At these points the gravitational pulls are in balance. Any infinitesimal body at any point of the Lagrangian points would be held there without getting pulled closer to either of massive bodies. $L_{1,2,3,}$ are collinear with the line joining the two massive bodies while the triangular points $L_{4,5}$ are found 60^0 ahead of and behind the less massive body along its orbit. $L_{4,5}$ are forming equilateral triangles with two massive bodies, see Fig. 1. The restricted three body problem (RTBP in brief) is now defined as a system consisting of two massive bodies, the primaries, revolving in a circular orbits around their centre of mass, and a third body of infinitesimally small mass which moves in the primaries orbital plane.

The post-Newtonian deviations of the triangular Lagrangian points from their classical positions in a fixed frame of reference for the first time, but without explicitly stating the equations of motion is computed[1].

The relativistic RTBP in rotating coordinates is treated[2]. He derived the Lagrangian of the system and the deviations of the triangular points as well.

The triangular points are stable in the linear system for the values of the mass ratio in the interval $(0, \mu)$ where $\mu = 0.038521$ is the Routhian value, is shown[3].

The global stability of these points has been studied by several authors: e.g. [4], [5], [6], and [7], their final conclusion is that in the planar case the triangular points $L_{4,5}$ are always stable within some domain of mass ratio.

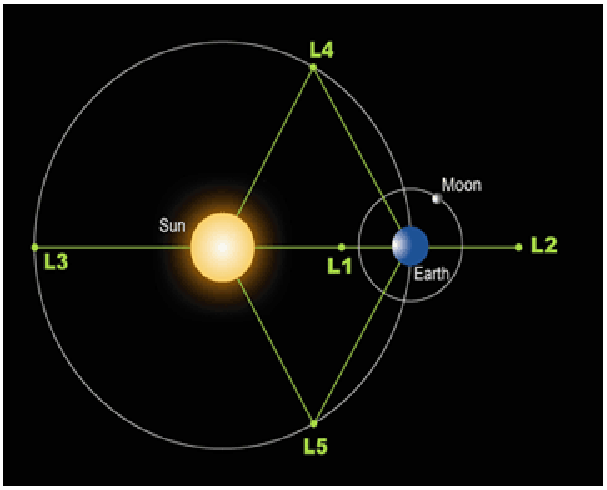

Fig. 1: The five Lagrangian points in RTBP (Sun-Earth system)

The equation of motion and the deviations of all Lagrangian points are derived directly from the equations of motion[8] .

The existence stability of the triangular points $L_{4,5}$ in the relativistic RTBP is studied [9], they concluded that $L_{4,5}$ always unstable in the whole range $0 \le \mu \le 0.5$ in contrast to the previous results of the classical RTBP where they are stable for $0 \le \mu \le \mu_0$, where μ is the mass ratio and $\mu_0 = 0.03852$.

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Due to this abnormal result obtained by [9], the location of the triangular libration points $L_{4,5}$ are recomputed and their stability are re-investigated to assure or to refuse or even to modify the above mentioned result.

EQUATIONS OF MOTION

The equations of motion of the infinitesimal mass in the relativistic RTBP in a synodic frame of reference (ξ,η), in the primaries coordinates on the x-axis $(-\mu,0),(1-\mu,0)$ are kept fixed and the origin at the centre of mass, are given by[10],

$$\left.\begin{aligned}\ddot{\xi}-2n\dot{\eta}&=\frac{\partial U}{\partial \xi}-\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{\xi}}\right),\\ \ddot{\eta}+2n\dot{\xi}&=\frac{\partial U}{\partial \eta}-\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{\eta}}\right).\end{aligned}\right\} \quad (1)$$

where U, is the potential–like function of the relativistic RTBP, which can be written as composed of two components, namely the classical RTBP potential U_C and the relativistic correction U_r;

$$U=U_c+U_r \quad (2)$$

where U_c and U_r are given by

$$U_c=\frac{r^2}{2}+\frac{1-\mu}{r_1}+\frac{\mu}{r_2} \quad (3)$$

and

$$\begin{aligned}U_r=&\frac{r^2}{2c^2}(\mu(1-\mu)-3)+\frac{1}{8c^2}((\xi+\dot{\eta})^2+\\&(\eta-\dot{\xi})^2)^2+\frac{3}{2c^2}\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right)((\xi+\dot{\eta})^2+\\&(\eta-\dot{\xi})^2)-\frac{1}{2c^2}\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right)^2-\frac{\mu(1-\mu)}{2c^2}\times\\&\left(\frac{1}{r_1}+\left(\frac{1}{r_1}-\frac{1}{r_2}\right)(1-3\mu-7\xi-8\dot{\eta})+\right.\\&\left.\eta^2\left(\frac{\mu}{r_1^3}+\frac{1-\mu}{r_2^3}\right)\right)\end{aligned} \quad (4)$$

with

$$\left.\begin{aligned}&n=1+\frac{1}{2c^2}(\mu(1-\mu)-3),\ r=\sqrt{(\xi^2+\eta^2)},\\&r_1=\sqrt{(\xi+\mu)^2+\eta^2},\ r_2=\sqrt{(\xi+\mu-1)^2+\eta^2}.\end{aligned}\right\} \quad (5)$$

LOCATIONS OF THE TRIANGULAR POINTS

The libration points are obtained from equations of motion after setting $\ddot{\xi}=\ddot{\eta}=\dot{\xi}=\dot{\eta}=0$. These points represent particular solutions of equations of motion

$$\left.\begin{aligned}\frac{\partial U}{\partial \xi}&=\frac{\partial U_c}{\partial \xi}+\frac{\partial U_r}{\partial \xi}=0,\\ \frac{\partial U}{\partial \eta}&=\frac{\partial U_c}{\partial \eta}+\frac{\partial U_r}{\partial \eta}=0.\end{aligned}\right\} \quad (6)$$

with $\dot{\xi}=\dot{\eta}=0$.

The explicit formulas are

$$\begin{aligned}\frac{\partial U}{\partial \xi}=&\left(\xi-\frac{(1-\mu)(\xi+\mu)}{r_1^3}-\frac{\mu(\xi+\mu-1)}{r_2^3}\right)+\\&\frac{1}{c^2}\left\{(\mu-\mu^2-3)\xi+\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right)\times\right.\\&\left(\frac{(1-\mu)(\xi+\mu)}{r_1^3}+\frac{\mu(\xi+\mu-1)}{r_2^3}\right)+\frac{1}{2}(\eta^2+\xi^2)\,\xi-\\&\frac{3}{2}\left(\frac{(1-\mu)(\xi+\mu)}{r_1^3}+\frac{\mu(\xi+\mu-1)}{r_2^3}\right)(\eta^2+\xi^2)+\\&3\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right)\xi-\frac{1}{2}\mu(1-\mu)\left[-\frac{1}{r_1^3}(\xi+\mu)+\right.\\&\left(\frac{(\xi+\mu)}{r_1^3}-\frac{(\xi+\mu-1)}{r_2^3}\right)(-1+3\mu+7\xi)-7\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\\&\left.\left.-3\eta^2\left(\frac{\mu(\xi+\mu)}{r_1^5}+\frac{(1-\mu)(\xi+\mu-1)}{r_2^5}\right]\right\}=0\end{aligned} \quad (7)$$

and

$$\begin{aligned}\frac{\partial U}{\partial \eta}=&\eta-\frac{(1-\mu)}{r_1^3}\eta-\frac{\mu}{r_2^3}\eta+\frac{1}{c^2}\left\{(\mu-\mu^2-3)\eta+\right.\\&\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right)\left(\frac{(1-\mu)}{r_1^3}+\frac{\mu}{r_2^3}\right)\eta+\frac{1}{2}(\xi^2+\eta^2)\eta-\\&\frac{3}{2}(\xi^2+\eta^2)\left(\frac{(1-\mu)}{r_1^3}+\frac{\mu}{r_2^3}\right)\eta+3\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right)\eta+\\&\frac{\mu(1-\mu)}{2}\eta\left[\left(-\frac{1}{r_1^3}+\frac{1}{r_2^3}\right)(-1+3\mu+7\xi)+\right.\\&\left.\left.3\left(\frac{(1-\mu)}{r_2^5}+\frac{\mu}{r_1^5}\right)\eta^2+\frac{1}{r_1^3}-2\left(\frac{\mu}{r_1^3}+\frac{1-\mu}{r_2^3}\right)\right]\right\}=0\end{aligned} \quad (8)$$

Since $\frac{1}{c^2}<<1$ and the solution of the classical RTBP is $r_1=r_2=1$, then it may be reasonable in our case to assume that positions of the equilibrium points $L_{4,5}$ are the same as given by classical RTBP but perturbed due to the inclusion of the relativistic correction by quantities ($\varepsilon_{1,2}\equiv o(\frac{1}{c^2})$)

$$r_1=(1+\varepsilon_1),\ r_2=(1+\varepsilon_2). \quad (9)$$

substituting in the second set of equations (5) and solving for ξ and η up to the first order in the small quantities ε_1 and ε_2 we get

$$\left.\begin{aligned}\xi &= (\varepsilon_1 - \varepsilon_2 + \frac{1-2\mu}{2}),\\ \eta &= \pm\left[\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{3}(\varepsilon_1+\varepsilon_2)\right].\end{aligned}\right\} \quad (10)$$

substituting the values of r_1, r_2, ξ and η into equations (7) and (8) and retaining only the first order terms of the relativistic correction yields

$$\left.\begin{aligned}(1-\mu)\varepsilon_1 - \mu\varepsilon_2 - \frac{3}{8c^2}\mu(1-2\mu)(1-\mu) &= 0,\\ (1-\mu)\varepsilon_1 + \mu\varepsilon_2 + \frac{7}{8c^2}\mu\,(1-\mu) &= 0.\end{aligned}\right\} \quad (11)$$

which represent two simultaneous equations in $\varepsilon_1, \varepsilon_2$ their solution gives

$$\left.\begin{aligned}\varepsilon_1 &= -\frac{1}{8c^2}\mu(2+3\mu),\\ \varepsilon_2 &= -\frac{1}{8c^2}(1-\mu)(5-3\mu).\end{aligned}\right\} \quad (12)$$

substituting the values of $\varepsilon_1, \varepsilon_2$ into equation (10) yields the coordinates of the triangular points $L_{4,5}$

$$\left.\begin{aligned}\xi &= \frac{(1-2\mu)}{2}(1+\frac{5}{4c^2}),\\ \eta &= \pm\frac{\sqrt{3}}{2}(1-\frac{1}{12c^2}(6\mu^2-6\mu+5)).\end{aligned}\right\} \quad (13)$$

THE STABILITY OF $L_{4,5}$

Due to the perturbations induced by the relativistic corrections the position of the infinitesimal body would be displaced a little from the equilibrium point. If the resultant motion of the particle is a rapid departure from the vicinity of the point we can call such a position of equilibrium point an unstable one, if however the partials merely oscillates about the equilibrium point, it is said to be a stable position.

To examine the stability of the orbits in the vicinity of the liberation points the equations of motion are linearized around an equilibrium point with coordinate $(\xi_o, \eta_o, \dot{\xi}_o = \dot{\eta}_o = 0)$

$$\left.\begin{aligned}\ddot{\xi}_o - 2n\dot{\eta}_o &= (\frac{\partial U}{\partial \xi})_{\xi=\xi_o},\\ \ddot{\eta}_o + 2n\dot{\xi}_o &= (\frac{\partial U}{\partial \xi})_{\eta=\eta_o}.\end{aligned}\right\} \quad (14)$$

The subscript o indicates evaluation for $\xi = \xi_o$ and $\eta = \eta_o$. Now if equations in (1) are evaluated at $\xi = \xi_o + \xi_1$ and $\eta = \eta_o + \eta_1$, one can write

$$\left.\begin{aligned}\ddot{\xi}_o + \ddot{\xi}_1 - 2n(\dot{\eta}_o + \dot{\eta}_1) &= U_\xi + (U_{\xi\xi})\xi_1 + (U_{\xi\eta})\eta_1 + ...,\\ \ddot{\eta}_o + \ddot{\eta}_1 + 2n(\dot{\xi}_o + \dot{\xi}_1) &= U_\eta + (U_{\eta\xi})\xi_1 + (U_{\eta\eta})\eta_1 + ...\end{aligned}\right\} \quad (15)$$

which can be rewritten as

$$\left.\begin{aligned}\ddot{\xi}_1 - 2n\dot{\eta}_1 &= (U_{\xi\xi})\xi_1 + (U_{\xi\eta})\eta_1 + ...,\\ \ddot{\eta}_1 + 2n\dot{\xi}_1 &= (U_{\eta\xi})\xi_1 + (U_{\eta\eta})\eta_1 + ...\end{aligned}\right\} \quad (16)$$

which are linear differential equations with constant coefficients so long as only first order terms are retained? Let a solution of the problem be

$$\xi_1 = A\exp(\sigma t),\ \eta_1 = B\exp(\sigma t) \quad (17)$$

where $A, B,$ and σ are constants. To find the expressions for A and B, equations (16) can be rewritten, using the suggested solution, as

$$\left.\begin{aligned}(\sigma^2 - U_{\xi\xi})A - (2\sigma + U_{\xi\eta})B &= 0,\\ (2\sigma + U_{\xi\eta})A + (\sigma^2 - U_{\eta\eta})B &= 0.\end{aligned}\right\}$$

(18) which can be written in matrix notations as

$$\begin{pmatrix}(\sigma^2 - U_{\xi\xi}) & -(2\sigma + U_{\xi\eta})\\ (2\sigma + U_{\xi\eta}) & (\sigma^2 - U_{\eta\eta})\end{pmatrix}\begin{pmatrix}A\\ B\end{pmatrix} = 0. \quad (19)$$

This system has non trivial solution if

$$\begin{vmatrix}(\sigma^2 - U_{\xi\xi}) & -(2\sigma + U_{\xi\eta})\\ (2\sigma + U_{\xi\eta}) & (\sigma^2 - U_{\eta\eta})\end{vmatrix} = 0 \quad (20)$$

expanding the determinate yields

$$\sigma^4 + (4 - U_{\xi\xi} - U_{\eta\eta})\sigma^2 + U_{\xi\xi}U_{\eta\eta} - U_{\xi\eta}^2 = 0 \quad (21)$$

where σ is the root of the characteristic determinate, and

$$U_{\xi\eta} \equiv (\frac{\partial^2 U}{\partial\xi\partial\eta})_{\xi=\xi_0, \eta=\eta_0}$$

Evaluating the partial derivatives included in equation (21), and neglecting the terms of orders $(c^{-n}, n \geq 3)$, now the characteristic equation (21) becomes

$$\sigma^4 + (1+\frac{A}{c^2})\sigma^2 + \frac{27}{4}\mu(1-\mu) + \frac{B}{c^2} = 0 \quad (22)$$

By substituting $\sigma = i\omega$ in the equation (22), we get

$$\omega^4 - (1+\frac{A}{c^2})\omega^2 + \frac{27}{4}\mu(1-\mu) + \frac{B}{c^2} = 0 \quad (23)$$

where

$$A = \left\{\frac{33}{4} - \frac{189}{16}\mu(1-\mu)\right\},$$

$$B = \left\{-\frac{7281}{256}\mu + \frac{10521}{256}\mu^2 - \frac{405}{16}\mu^3 + \frac{405}{32}\mu^4\right\}.$$

SOLUTION OF EQUATION (22)

Similarly as in the nonrelativistic RTBP equation (22) is biquadratic in σ. This is possible only for the simplistic 4×4 matrices and reflects the simplistic structure of the phase space of the relativistic RTBP which is also a consequence of the fact that the equations of motion of the relativistic RTBP are derived from a Lagrangian.

In the case of linear stability, none of the eigenvalues' real parts is positive, i.e. if the equation (22) has two negative non-equal roots. The solution is given by

$$\sigma_{1,2}^2 = -\frac{1}{2}(1+\frac{A}{c^2}) \pm \frac{1}{2}\sqrt{\left(1-27\mu+2(\frac{A}{c^2})+(\frac{A}{c^2})^2-4(\frac{B}{c^2})\right)} \qquad (24)$$

The following three cases for the solutions $\sigma_{1,2}^2$ are possible:-

(1) The first case: when $\sigma_{1,2}^2$ is complex with non-vanishing imaginary part: $\sigma_{1,2}^2 = x \pm iy,\ y \neq 0$, it follows that

$$\sigma = X + iY \qquad (25)$$

with

$$X = \pm\frac{1}{\sqrt{2}}\sqrt{x+\sqrt{x^2+y^2}}\ ,$$

$$Y = \pm\frac{y}{\sqrt{2(x+\sqrt{x^2+y^2})}}\ .$$

Accordingly, there is a solution σ with positive real part thus instability.

(2) The second case: when $\sigma_{1,2}^2$ is real and positive. In this case one of the eigenvalues $\sigma = \pm\sqrt{\sigma_{1,2}^2}$, namely the positive root induces instability.

(3) The third case: when $\sigma_{1,2}^2$ is real and negative. In this case two purely imaginary roots $\sigma = \pm i\sqrt{|\sigma_{1,2}^2|}$ exist, which leads to oscillatory stable solutions. Hence in the following section real $\sigma_{1,2}^2 < 0$ will be investigated.

CRITICAL MASS RATIO FOR STABLE $L_{4,5}$

A sufficient condition for real $\sigma_{1,2}^2$ is

$$1-27\mu+27\mu^2+2(\frac{A}{c^2})+(\frac{A}{c^2})^2-4(\frac{B}{c^2}) = F(\mu) \geq 0 \qquad (26)$$

Since in the μ interval $[0,\frac{1}{2}]$, the expression

$$\frac{\partial}{\partial\mu}(F) = \frac{9}{64}(1-2\mu)[-192+\frac{1}{c^2}(641-720\mu+720\mu^2] \qquad (27)$$

has a zero only for $\mu = 0.5$, F is a monotonous function of $\mu \in [0,0.5]$. As $F|_{\mu=0} > 0$ and $F|_{\mu=0.5} < 0$, there is a unique solution of $F(\mu) = 0$ in this interval.

The critical mass ratio $\mu_{crit.}$ for the stable triangular points can be assumed as composed of two parts as,

$$\mu_{crit} = \mu_c + \frac{1}{c^2}\mu_r \qquad (28)$$

where μ_c, μ_r are the contributions due to the classical and relativistic RTBP respectively.

Substituting from (28) into the condition (26) leads to the conditions

$$1-27\mu_c(1-\mu_c) = 0 \qquad (29)$$

and

$$\left(\frac{33}{2}-27\mu_r(1-2\mu_c)+\frac{5769}{64}\mu_c-\frac{9009}{64}\mu_c^2+\frac{405}{4}\mu_c^3-\frac{405}{8}\mu_c^4\right) = 0 \qquad (30)$$

The conditions (29) and (30) represent two simultaneous equations in two variables, namely μ_c and μ_r. Their solution yields the mass ratios as

$$\mu_c = (\frac{1}{2}-\frac{1}{18}\sqrt{69}),\quad \mu_r = \frac{11387}{119232}\sqrt{69}\ .$$

According to equation (24), $\sigma_{1,2}^2$ is a monotonous faction of μ as well as F is. At the points of interest $\mu = 0$ and $\mu = \mu_{crit.}$ it takes values less than or equal to zero, so that for $\mu \in [0,\mu_{crit.}]$ each of the eigenvalues σ has a zero real part. Finally the critical mass ratio for stable triangular points in the relativistic RTBP is given by

$$\mu_{crit.} = (\frac{1}{2}-\frac{1}{18}\sqrt{69}) + \frac{1}{c^2}\frac{11387}{119232}\sqrt{69} \qquad (31)$$

In the non dimensional units of the relativistic RTBP the parameter $\frac{1}{c^2}$ is related to the ration of the total mass of the system M and the separation of the primaries R such that $\frac{1}{c^2}$ becomes proportional to $\frac{M}{R}$ by

$$\frac{1}{c^2} = 9.970154\times10^{-9}(\frac{M}{R})$$

where M and R are expressed in solar masses and astronomical unites, respectively. Now the critical mass ratio is given by

$$\mu_{crit.} = (\frac{1}{2}-\frac{1}{18}\sqrt{69})+7.9094\times10^{-9}(\frac{M}{R}) \qquad (31)$$

Thus in the relativistic RTBP the critical mass ratio $\mu_{crit.}$ is shifted towards the greater values for increasing $\frac{M}{R}$.

RESULTS AND CONCLUSION

To investigate the region of stability and instability the roots ω's given by equation (23) are plotted against different values of mass ratio μ, by using the Mathematica 5 software. See Figs. 2-6. As is clear from these Figures, the stability region is shifted with about $\Delta\mu = -1.209\times10^{-4}$. This shift is due to the inclusion of the relativistic corrections in the restricted three body problem. A new value of the maximum mass ratio is obtained, i.e. the triangular points $L_{4,5}$ are stable for the values of the mass ratio in the interval $(0, \mu_0)$ where $\mu_0 = 0.03840$ instead of the classical Routhian value $\mu_0 = 0.03852$.

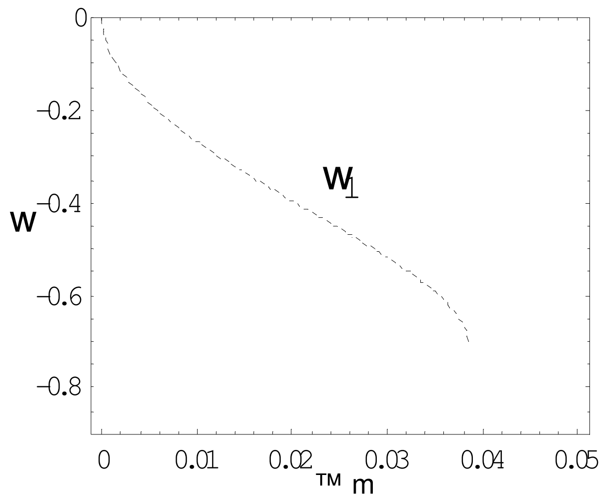

Fig.2: The real values of the root ω_1 versus μ for the characteristic equation (22)

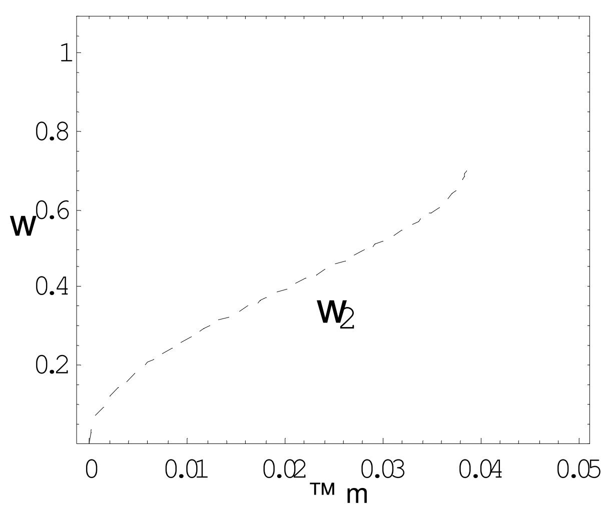

Fig.3: The real values of the root ω_2 versus μ for the characteristic equation (22)

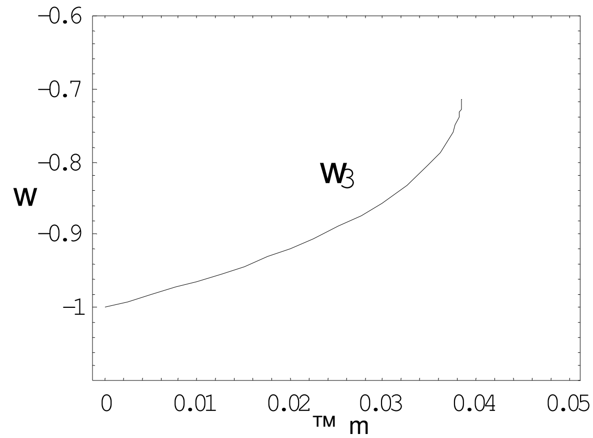

Fig.4: The real values of the root ω_3 versus μ for the characteristic equation (22)

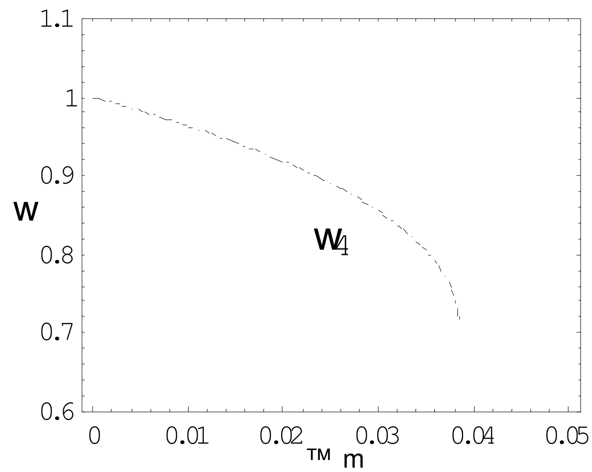

Fig.5: The real values of the root ω_4 versus μ for the characteristic equation (22)

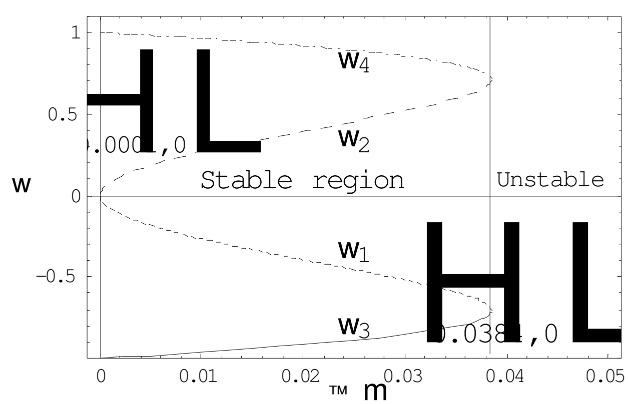

Fig.6: The stable and unstable region of the relativistic corrections in the restricted three body problem for $L_{4,5}$

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REFERENCES

1. Krefetz, E., 1967. Restricted Three-Body problem in the Post Newtonian Approximation. J. Astron., 72:471-473.
2. Contopoulos, G. 1976. In Memorian D. Eginitis, ed. By D. Kotsakis, 159.
3. Szebehely, V., 1967. Theory of Orbits. Academic Press, New York.
4. Leontovich, A.M., 1962. On the Stability of the restricted problem of three bodies. Soviet math. Dokl., 3:425-428.
5. Deprit, A. and Deprit-Bartholome, 1967. Stability of Triangular Lagrangian Points. J. The Astron., 72:173-179.
6. Markeev, A.P., 1969. On the stability of the Triangular Libration points in the Circular Bounded Three Body Problem. Appl. Math., 33: 105-110.
7. Rusmann, H., 1979. On the stability of an equilibrium Solution in the Hamiltonian System of Two Degrees of Freedom. In Instabilities in Dynamical Systems ed. V. Szebehely, NATO, ASI, Italy, pp303-305.
8. Maindl T.I. and R. Dvorka, 1994. On the dynamic of the relativistic restricted three-body problem. Aston. and Astrophys., 290:335-339.
9. Bhatnagar, K.B. and Hallan, P.P., 1998. Existence and stability of $L_{4,5}$ in the relativistic restricted three body problem. Celest. Mech., 69: 271-281.
10. Brumberg, V.G., 1972. Relativistic Celestial Mechanics. Nauka, Moscow.